International Mathematical Forum, Vol. 7, 2012, no. 19, 925 - 934

Spectral Compactification of a Ring

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Abstract

It is well known that if a ring has an identity then its prime spectrum is compact and if it does not, the spectrum could be non-compact. There are two standard methods to adjoin an identity to a commutative ring of non-zero characteristic n. Through these methods we obtain two unitary rings: one of characteristic n and the other of characteristic zero. If the spectrum of the original ring is non-compact, then the two spectra of the new rings contain a compactification by finite points of the original spectrum. Although the spectra are different, the two compactifications are homeomorphic.

Mathematics Subject Classification: Primary 54B35. Secondary 54D35, 13A15

Keywords: Prime spectrum, identity, characteristic, compact space, compactification

1 Introduction.

Throughout this paper a ring is always assumed to be commutative and not necessarily unitary. A $K-$algebra is always associative and commutative. Compact spaces are not assumed to be Hausdorff.

The prime spectrum of a ring A is the set of its prime ideals endowed with the Zarisky topology, whose basic open sets are given by

$$D(a) = \{I : I \text{ is a prime ideal of } A \text{ and } a \notin I\},$$

where $a \in A$. We denote this space by $Spec(A)$, as usual. It is known that the prime spectrum of a commutative ring with identity is a compact topological space (see [4]); however, when the ring does not have an identity this space can or cannot be compact. To avoid confusion with compact topological rings, we say that a ring is *spectrally compact* if its prime spectrum is a compact topological space.

Given a non spectrally compact ring, we are interested in the relationship between its prime spectrum and the prime spectrum of the ring obtained by adjoining an identity. Although this relationship appears to be quite natural, there are few references about it. Nevertheless, there are many papers that study the relationship between topological properties of the spectrum and algebraic properties of the ring. Hochster [9] characterizes the topological spaces that are homeomorphic to the spectra of commutative rings with identity, the spectral spaces. A-spectral spaces, spaces whose Alexandroff compactification is a spectral space, are studied in [5]. In [1], Acosta and Galeano show that in the case of Boolean rings without identity, the spectrum obtained by adjoining an identity (as a $\mathbb{Z}_2-$algebra) is the Alexandroff compactification of the original space and later in [6], Camacho shows that this result cannot be extended even in the case of non-compact rings of characteristic two.

Most authors assume that their rings are unitary, although sometimes they do not use this fact in their proofs. In [2], Anderson reviews several papers that study rings without identity and he shows that sometimes the absence of identity is irrelevant in the proofs, while in other cases, the lack of identity is not resolved by simply adjoining one, but actually requires a careful treatment. Though there are different ways to adjoin an identity to a ring, we use the method described in [12] for $K-$algebras, which is a generalization of the method that we find in [7] and [10], for the cases in which K is $\mathbb{Z}$, $\mathbb{Z}_n$ or $\mathbb{Q}$.

In the first section of this paper we show that the prime spectrum of an ideal of a ring is always a subspace of the prime spectrum of the ring. In the second section we use the previous result to show that the prime spectrum of the ring obtained by adjoining an identity to a non spectrally compact ring contains a compactification of the prime spectrum of the original ring. In the third section we show that in the case of rings of characteristic not equal to zero, though it is possible to adjoin an identity in at least two different ways, the compactifications obtained by the methods explained in the preceding section, are homeomorphic. Finally we build an example that illustrates that when the compactification obtained by the process of adjoining an identity is by only one point, it is not necessarily the Alexandroff compactification. We are omiting some proofs because they are simple.

2 On prime spectra.

In this section we present some results that allow us to relate the prime spectrum of a ring and the spectra of its ideals. We conclude that the prime spectrum of an ideal of a ring is always a subspace of the prime spectrum of the ring.

Theorem 2.1. *Let R be a ring, S an ideal of R and I an ideal of S. We define:*

$$\psi(I) = \{a \in R : aS \subseteq I\}.$$

Then:

1. *the set $\psi(I)$ is an ideal of R;*
2. *if I is a prime ideal of S, then $\psi(I)$ is a prime ideal of R.*

This theorem allows us to consider ψ as a function from $Spec(S)$ to $Spec(R)$, which has very interesting properties as we see below. When S is an ideal of the ring R, we denote $Spec_S(R)$ the subspace of $Spec(R)$ of the prime ideals of R which do not contain S.

Theorem 2.2. *Let R be a ring and S an ideal of R. The function $\psi : Spec(S) \to Spec_S(R)$ defined in the previous theorem, is a homeomorphism and its inverse is $\varphi : Spec_S(R) \to Spec(S) : J \mapsto J \cap S$.*

Proof. It is immediate that φ is well defined and $\psi^{-1} = \varphi$. In addition, if $a \in R$ then $\psi^{-1}(D(a)) = \bigcup_{x \in S} D(ax)$, therefore ψ is continuous. Also, if $b \in S$ it is clear that $\varphi^{-1}(D(b)) = D(b) \cap Spec_S(R)$, then φ is continuous. □

Remark 2.3. *The ideal $\psi(I)$ is denoted $I : S$ in [8] and $[I : S]_R$ in [3]. In [8] Gilmer proves the second part of Theorem 2.1 and also that $\varphi(\psi(I)) = I$.*

Given these results, we can state that the prime spectrum of each ideal S of the ring R is a subspace of the prime spectrum of R.

3 Adjunction of identity vs compactification.

Let K be a ring with identity and A a $K-$algebra. We denote $U_K(A)$ or simply $U(A)$, where there is no danger of confusion, the set $A \times K$ endowed with addition defined componentwise, multiplication defined by

$$(a,\alpha)(b,\beta) = (ab + \beta a + \alpha b, \alpha\beta)$$

and product by elements of K defined by

$$\beta(a,\alpha) = (\beta a, \beta\alpha).$$

It is easy to check that $U(A)$ is a unitary $K-$algebra with identity $(0,1)$ and that we have the following universal property.

Proposition 3.1. *For each unitary $K-$algebra B and for each homomorphism of $K-$algebras $h : A \to B$ there exists a unique homomorphism of unitary $K-$algebras $\widetilde{h} : U(A) \to B$ such that $\widetilde{h} \circ i_A = h$, where $i_A : A \to U(A) : i_A(a) \mapsto (a,0)$.*

Let A_0 denote the set $A \times \{0\} \subset U(A)$. It is clear that A_0 is an ideal of $U(A)$. On the other hand, the homomorphism i_A identifies ring A with A_0 in $U(A)$. Then $Spec(A)$ is homeomorphic to $Spec_{A_0}(U(A))$ as a consequence of Theorem 2.2. Therefore in this process of adjoining an identity to a $K-$algebra, the original spectrum is always a topological subspace of the spectrum of the new ring.

Theorem 3.2. *If A is not spectrally compact then $Spec(U(A))$ contains a compactification of $Spec(A)$ by at most $|Spec(K)|$ points.*

Proof. We consider the map $U(A) \to K$ given by $(a,\alpha) \mapsto \alpha$. Since this is a surjective homomorphism, the prime ideals of $U(A)$ that contain A_0 are in bijective correspondence with the prime ideals of K. Since $U(A)$ has an identity, $Spec(U(A))$ is compact and thus the compactification of $Spec(A)$ is $\overline{Spec_{A_0}(U(A))}$ as a subspace of $Spec(U(A))$. □

Since every ring is a $\mathbb{Z}-$algebra and $Spec(\mathbb{Z})$ is countable, then for every non spectrally compact ring A, $Spec(U_{\mathbb{Z}}(A))$ contains a compactification of $Spec(A)$ by countably many points at most. The additional points in $Spec(U_{\mathbb{Z}}(A))$ correspond to the ideal A_0 and the ideals $A \times p\mathbb{Z}$, for each prime number p.

If K is a field and A is a non spectrally compact $K-$algebra, then $Spec(U_K(A))$ is a compactification of $Spec(A)$ by exactly one point. When A is a non spectrally compact ring of prime characteristic p, then A is a $\mathbb{Z}_p-$algebra and therefore $Spec\left(U_{\mathbb{Z}_p}(A)\right)$ is a compactification of the prime spectrum of A by one

point. In particular, if B is a Boolean ring without identity, $Spec\,(U_{\mathbb{Z}_2}(B))$ is a compactification of $Spec\,(B)$ by one point. In [1], Acosta and Galeano obtained that in this case, $Spec\,(U_{\mathbb{Z}_2}(B))$ is precisely the Alexandroff compactification of $Spec\,(B)$.
The next theorem states a sufficient condition for $Spec\,(U\,(A))$ to be a compactification of $Spec\,(A)$.

Theorem 3.3. *If there exists $I \in Spec(A)$ such that $I \times \{0\}$ is a prime ideal of $U(A)$, then $Spec_{A_0}(U\,(A))$ is a dense subspace of $Spec(U(A))$.*

Proof. It is sufficient to see that every prime ideal of $U(A)$ that contains A_0 is in the closure of $Spec_{A_0}(U\,(A))$. Let J be a prime ideal of $U(A)$ that contains A_0. Then $\bigcap Spec_{A_0}(U\,(A)) \subseteq I \times \{0\} \subseteq A_0 \subseteq J$ and therefore $J \in \overline{Spec_{A_0}(U\,(A))}$. □

Corollary 3.4. *If A is a non spectrally compact ring and there exists $I \in Spec(A)$ such that $I \times \{0\}$ is a prime ideal of $U(A)$, then $Spec(U(A))$ is a compactification of $Spec(A)$ by $|Spec(K)|$ points.*

In Section 5 we show an example of this situation.

4 Rings of characteristic $n \neq 0$.

Let A be a ring of characteristic $n \neq 0$. In this section we assume that $n = p_1^{\alpha_1}...p_m^{\alpha_m}$, where the p_i are distinct prime numbers. As A is a $\mathbb{Z}_n-$algebra, we will denote by $U_n(A)$ the $\mathbb{Z}_n-$algebra $U_{\mathbb{Z}_n}(A)$. The prime ideals of $U_n(A)$ that contain A_0 are of the form $A \times p_i\mathbb{Z}_n$, for $i = 1, ..., m$, one for each prime divisor of n. If A is non spectrally compact, then $Spec\,(U_n(A))$ contains a compactification of $Spec\,(A)$ by at most m points. In particular, if $n = p^\alpha$ then $Spec\,(U_n(A))$ is a compactification of $Spec\,(A)$ by exactly one point.

The following example shows that the obtained compactification is not necessarily by m points.

Example 4.1. *Let B be a Boolean ring without identity. The ring $A = B \times \mathbb{Z}_3$ is of characteristic 6, without identity and it has non-compact prime spectrum. Therefore $Spec\,(U_6(A))$ contains a compactification of $Spec\,(A)$ by at most two points: $A \times 2\mathbb{Z}_6$ and $A \times 3\mathbb{Z}_6$.*
Since $A \times 3\mathbb{Z}_6$ is an isolated point in $Spec\,(U_6(A))$, $Spec\,(A)$ is not dense in $Spec\,(U_6(A))$. Hence $Spec\,(U_6(A))$ is not a compactification of $Spec\,(A)$ by two points, but it contains a compactification of $Spec\,(A)$ by one point.

Every commutative ring A of characteristic $n \neq 0$ is also a $\mathbb{Z}-$algebra. Therefore $Spec\,(U_{\mathbb{Z}}(A))$ contains a compactification of $Spec\,(A)$ by at most countably many points. In this section we study the relationship between the two compactifications of $Spec\,(A)$, one contained in $Spec\,(U_n\,(A))$, and the other contained in $Spec\,(U_{\mathbb{Z}}\,(A))$.

4.1 Characteristic and prime ideals.

We now mention some properties that relate the prime ideals and the characteristic of the quotient rings determined by them.

Definition 4.2. *Let A be a ring of characteristic $n \neq 0$. Consider the function $c : Spec\,(A) \to \mathbb{N} : I \mapsto char\,(A/I)$. We say that the natural number $c\,(I)$ is the class of the prime ideal I.*

This definition allows us to partition the set of prime ideals according to their class. We now present some properties of the function of class c. Notice that $c\,(I)$ is always a prime number.

Proposition 4.3. *Let A be a ring and I, I_1, I_2 prime ideals of A. Then*

1. $c\,(I)$ divides the characteristic of A;
2. $c\,(I)\,A \subseteq I$;
3. if p is a prime number then $pA \subseteq I$ if and only if $c\,(I) = p$;
4. if $I_1 \subseteq I_2$ then $c\,(I_1) = c\,(I_2)$;
5. if $J \in Spec\,(U_{\mathbb{Z}}\,(A))$ and p is a prime number, then $c\,(J) = p$ if and only if $(0, p) \in J$.

Proof.

1. Let $a+I$ be an element of A/I. Therefore $n\,(a+I) = na+I = 0+I = I$. This implies that $c\,(I)$ is a prime number p less than n. Suppose that $p \neq p_i$ for each $i = 1, ..., m$. Then for each $i = 1, ..., m$, there exists $a_i \in A$ such that $p_i\,(a_i + I) \neq I$. Thus, $\prod_{i=1}^{m} (p_i\,(a_i + I))^{\alpha_i} = n\,(a_1^{\alpha_1} ... a_m^{\alpha_m} + I) = I$, because $char(A) = n$. But this is a contradiction since A/I is a ring without zero divisors. Therefore $char(A/I) = p_i$, for some $i = 1, ..., m$.
2. Assume that $c\,(I) = p$, with p a prime number. Let $a \in A$, then $I = p\,(a+I) = pa + I$, and $pa \in I$.
3. By the previous property, it is sufficient to assume that $pA \subseteq I$. Let $a \in A$. If $pa \in I$ then $p\,(a+I) = pa + I = I$, so $char\,(A/I) = p$.
4. and 5. are straightforward.

□

Proposition 4.4. *Let J be a prime ideal of $U_{\mathbb{Z}}(A)$ such that $\varphi(J) = I$ is a prime ideal of A. Then $c(J) = c(I)$.*

Proof. Let $p = c(I)$ and $a \in A - I$. Thus, $pa \in I$. Then $(a,0)(0,p) = (pa,0) \in J$ and $(0,p) \in J$. Therefore the class of J is also p. □

4.2 Relationship between $U_n(A)$ and $U_{\mathbb{Z}}(A)$.

We denote by $\theta : \mathbb{Z} \to \mathbb{Z}_n$ the canonical quotient homomorphism. We define the function $\rho : U_{\mathbb{Z}}(A) \to U_n(A) : (a,\alpha) \mapsto (a,\theta(\alpha))$, which clearly is a homomorphism of unitary rings. As ρ acts as the identity on A_0, then, by the universal property of the functor $U_{\mathbb{Z}}$, ρ is the unique homomorphism from $U_{\mathbb{Z}}(A)$ to $U_n(A)$ with this property. Since ρ is surjective, $U_n(A)$ is a quotient of $U_{\mathbb{Z}}(A)$; more precisely $U_n(A) \cong U_{\mathbb{Z}}(A) / (\{0\} \times n\mathbb{Z})$.
On the other hand $Spec(\rho) : Spec(U_n(A)) \to Spec(U_{\mathbb{Z}}(A)) : J \mapsto \rho^{-1}(J)$ is a continuous function. We denote by ψ_n and ψ_0 the inclusions from $Spec(A)$ to $Spec(U_n(A))$ and to $Spec(U_{\mathbb{Z}}(A))$, respectively.

Proposition 4.5. *$Spec(\rho)$ is injective and $\psi_0 = Spec(\rho) \circ \psi_n$.*

Proof. If $p \in \{p_1, ..., p_m\}$ then $Spec(\rho)(A \times p\mathbb{Z}_n) = \rho^{-1}(A \times p\mathbb{Z}_n) = A \times p\mathbb{Z}$. Let $I \in Spec(A)$. It is clear that $(a,\alpha) \in Spec(\rho)(\psi_n(I))$ is equivalent to $\rho(a,\alpha) \in \psi_n(I)$. So $(a,\theta(\alpha)) A_0 \subseteq I \times \{0\}$ and $(a,\alpha) \in \psi_0(I)$. □

Theorem 4.6. *The function $Spec(\rho)$ is a homeomorphism between $Spec(U_n(A))$ and its image.*

Proof. We denote by $\underline{D}(a,\alpha)$ the basic open sets of the image of $Spec(\rho)$ as a subspace of $Spec(U_{\mathbb{Z}}(A))$. To see that $Spec(\rho)$ is an open function onto its image it is sufficient to observe that for all $(a,\alpha) \in U_n(A)$, $Spec(\rho)(D(a,\alpha)) = \underline{\bigcup_{\lambda \in \mathbb{Z}} D}(a, \alpha + \lambda n)$. □

Corollary 4.7. *If A is a non spectrally compact ring of characteristic $n = p_1^{\alpha_1} ... p_m^{\alpha_m}$, then $Spec(U_{\mathbb{Z}}(A))$ and $Spec(U_n(A))$ contain homeomorphic compactifications of $Spec(A)$ by t points, for some $t \in \{1, ..., m\}$.*

Observe that A_0 and $A \times p\mathbb{Z}$, where p is a prime number that does not divide n, are not elements of $\overline{Spec_{A_0}(U_{\mathbb{Z}}(A))}$. Then the additional points in $\overline{Spec_{A_0}(U_{\mathbb{Z}}(A))}$ are of the form $A \times p_i\mathbb{Z}$, where $i \in \{1, ..., m\}$.

5 On the compactification of a polynomial ring.

In this section we generalize to the context of $K-$algebras some results presented by Camacho in [6]. We give an example that shows that even though the compactification obtained by the method described in the previous sections is by one point, this is not necessarily the Alexandroff compactification. The interested reader can consult on the Alexandroff compactification in [11].

Given K a unitary ring and A a $K-$algebra without identity, we denote by $A\{x\}$ the $K-$algebra without identity with elements of the form

$$\sum_{i=0}^{n} a_i x^i + \sum_{j=1}^{m} \alpha_j x^j, \text{ where } a_i \in A, \alpha_j \in K$$

and by $A[x]$ the usual ring of polynomials in an indeterminate x, with coefficients in A. Thus $A[x]$ is a $K-$subalgebra of $A\{x\}$.

Theorem 5.1. *If K is an integral domain and A is a $K-$algebra without identity, then*

1. *$A[x]$ is a prime ideal of $A\{x\}$.*
2. *$A[x] \times \{0\}$ is a prime ideal of $U_K(A\{x\})$.*

Proof. It is sufficient to note that $A\{x\}/A[x] \cong xK[x]$ and $U_K(A\{x\})/(A[x] \times \{0\}) \cong U_K(A\{x\}/A[x]) \cong U_K(xK[x]) \cong K[x]$. □

The ideal generated by x in the ring $A\{x\}$ is the set

$$\langle x \rangle = \{xp(x) + \alpha x \in A\{x\} \mid p(x) \in A\{x\}, \alpha \in K\}$$

and its elements are precisely the polynomials of $A\{x\}$ with no independent term.

We now study some relations between the spaces $Spec(A)$ and $Spec(A\{x\})$. In order to avoid confusions we denote by $D\{p(x)\}$ the basic open sets of $Spec(A\{x\})$ and as usual, by $D(a)$ the basic open sets of the spectrum of A.

Proposition 5.2. *If K is a unitary ring and A is a $K-$algebra, then* $Spec(A\{x\}) = D\{x\} \cup \left(\bigcup_{a \in A} D\{a\} \right)$.

Proof. Let $J \in Spec(A\{x\}) \setminus D\{x\}$. We have that $\langle x \rangle \subseteq J$, hence all the polynomials with no independent term are elements of J. If $A \subseteq J$ we have that $J = A\{x\}$, which is absurd. Then there exists $a \in A - J$, so $J \in D\{a\}$ and therefore $J \in \bigcup_{a \in A} D\{a\}$. □

Theorem 5.3. *Let K be a unitary ring and A a $K-$algebra without identity. If $A\{x\}$ is spectrally compact then A is spectrally compact.*

Proof. Let A such that $A\{x\}$ is spectrally compact. By Proposition 5.2 there exist $a_1, a_2, ..., a_n \in A$ such that $Spec(A\{x\}) = D\{x\} \cup D\{a_1\} \cup ... \cup D\{a_n\}$. For each $I \in Spec(A)$ we define $\widetilde{I} = \{i + p(x) \in A\{x\} \mid i \in I, p(x) \in \langle x \rangle\}$. It is clear that $\widetilde{I} \in Spec(A\{x\})$ and $I \subseteq \widetilde{I}$. So, if $\widetilde{I} \in D\{a_i\}$ then $I \in D(a_i)$.
We conclude that $Spec(A) = D(a_1) \cup ... \cup D(a_n)$ and therefore $Spec(A)$ is compact. □

As a consequence of Corollary 3.4 and the previous observations, if A is a non spectrally compact $K-$algebra, then $Spec(U_K(A\{x\}))$ is a compactification of $Spec(A\{x\})$ by exactly $|Spec(K)|$ points. In particular, if K is a field then $Spec(U_K(A\{x\}))$ is a compactification of $Spec(A\{x\})$ by one point. The following theorem shows that this compactification is not the Alexandroff compactification.

Theorem 5.4. *If K is a field and A is a non spectrally compact $K-$algebra, then $Spec(U_K(A\{x\}))$ is not the Alexandroff compactification of $Spec(A\{x\})$.*

Proof. Let $V = \underset{a\in A}{\cup} D\{a\}$. We will see that $T = V \cup \{A\{x\} \times \{0\}\}$ is an open set of the Alexandroff compactification of $Spec(A\{x\})$ that is not an open set of $Spec(U_K(A\{x\}))$.
As $Spec(A\{x\}) = D\{x\} \cup V$ then $V^c \subseteq D\{x\}$. Since V is open and $D\{x\}$ is compact then V^c is compact, thus T is an open set of the Alexandroff compactification of $Spec(A\{x\})$.
To be precise, T in $Spec(U_K(A\{x\}))$ is
$\widetilde{T} = \psi(V) \cup \{A\{x\} \times \{0\}\} = \underset{a\in A}{\cup} D(a, 0) \cup \{A\{x\} \times \{0\}\}$.
Suppose that there exists $(p(x), \alpha) \in U_K(A\{x\})$ such that $A\{x\} \times \{0\} \in D(p(x), \alpha) \subseteq \widetilde{T}$. Then $(p(x), \alpha) \notin A\{x\} \times \{0\}$, thus $\alpha \neq 0$.
Call J the prime ideal $A[x] \times \{0\}$ of $U_K(A\{x\})$. Then $(p(x), \alpha) \notin J$, and $J \in D(p(x), \alpha)$. But $J \notin \widetilde{T}$, that is, $D(p(x), \alpha) \nsubseteq \widetilde{T}$. Indeed, if we suppose that $J \in \widetilde{T}$, there exists $a \in A$ such that $(a, 0) \notin J$, which is not possible because $A \times \{0\} \subseteq J$. Therefore $A\{x\} \times \{0\}$ is not an interior point of $\widetilde{T}$ and $\widetilde{T}$ is not an open set of $Spec(U_K(A\{x\}))$. □

Question 5.5. *How to describe algebraically the rings A of characteristic $n = p_1^{\alpha_1}...p_m^{\alpha_m}$ such that $Spec(U_n(A))$ is a compactification of $Spec(A)$ by m points?*

Question 5.6. *Let S be an ideal of the ring R. If R is spectrally compact then $X = \overline{Spec_S(R)}$ is a compactification of $Spec(S)$. Is X a spectral space?*

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Received: September, 2011