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Distribution for the Standard Eigenvalues of Quaternion Matrices

Shahid Qaisar

College of Mathematics and Statistics, Chongqing University
Chongqing, 401331, P. R. China
shahidqaisar90@yahoo.com

Limin Zou

College of Mathematics and Statistics, Chongqing University
Chongqing, 401331, P. R. China
limin-zou@163.com

Abstract

This paper aims to estimate and locate the standard eigenvalues of quaternion matrices. We prove that all the standard eigenvalues of a central closed matrix are located in one Geršgorin ball. We shall conclude the paper with some numerical examples which will show the effectiveness of our new results.

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1 Introduction

Quaternion was introduced by the Irish mathematician Hamilton (1805-1865) in 1843[1]. Hamilton probably never thought that one day in the future the quaternions he had invented would be used in computer graphics, control theory, physics, and mechanics.

In some differential equations problems involving the long-term stability of an oscillating system, one is sometimes interested in showing that the eigenvalues of a quaternion matrix all lie in the left half-plane. And sometimes in statistics or numerical analysis one needs to show that a Hermitian quaternion matrix is positive definite.

It is well known that the main obstacle in the study of quaternion matrices is the non -commutative multiplication of quaternions. So, it is not easy to study quaternion algebra problems. However, quaternion algebra theory is becoming more and more important in recent years, and the study of quaternion matrices is still developing because of the above application [2-4].

Throughout the paper, we use a set of notation and terminology. R and C are the set of the real and complex numbers, respectively. Let H be the normed vector space and skew field of real quaternions:

$$H = \{a = a_0 + a_1 i + a_2 j + a_3 k, a_0, a_1, a_2, a_3 \in R\}.$$

It is common to use H for quaternion in honor of the inventor, Hamilton. For any $a = a_0 + a_1 i + a_2 j + a_3 k \in H$, the conjugate of a is $\overline{a} = a^* = a_0 - a_1 i - a_2 j - a_3 k$ and the modulus of a is $|a| = \sqrt{a\overline{a}} = \sqrt{\overline{a}a} = (a_0^2 + a_1^2 + a_2^2 + a_3^2)^{1/2}$. Let $H^{n\times n}$ and $H^{n\times 1}$ be the collections of all $n \times n$ matrices with entries in H and n-column vectors, respectively. Let I_n and $H(n, u)$ be the collections of all $n \times n$ unit matrices with entries in H and quaternion unitary matrix, respectively. For $X \in H^{n\times 1}$,X^T is the transpose of X. If $X = (x_1, x_2 \ldots x_n)^T$, then $\bar{X} = (\bar{x_1}, \bar{x_2}, \ldots, \bar{x_n})^T$ is the conjugate of X and $X^* = (\bar{x_1}, \bar{x_2}, \ldots, \bar{x_n})$ is the conjugate transpose of X. The modulus of X is defined to be $|X| = \sqrt{X^* X}$. For an $n \times n$ matrix $A = (a_{ij})_{n\times n}$ $(a_{ij} \in H)$, the conjugate transpose of A is the $n \times n$ matrix $A^* = \bar{A}^T$.

The well-known Geršgorin theorem is one of fundamental theorems in complex matrix theory. It guarantees that all the eigenvalues of a matrix are located in the union of n Geršgorin disks. Recently, the Geršgorin theorem has been generalized to real quaternion division algebra [3, 5].

In this paper, we prove that all the standard eigenvalues of a central closed matrix are located in one Geršgorin ball. To do this, we need the following definitions and lemmas.

2 Some definitions and lemmas

Definition 2.1[6] Suppose that a is a given real quaternion and R is a positive real number. The set $G = \{z \in H : |z - a| \le R\}$ is said to be a Geršgorin ball with the center a and the radius$R$.

Definition 2.2 Let$A = (a_{ij})_{n\times n} \in H^{n\times n}$. The Frobinus norm of A is defined by

$$\|A\|_F = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2\right)^{\frac{1}{2}} = \sqrt{Tr\,(A^* A)}..$$

Lemma 2.3[4] Any $n \times n$ quaternion matrix A has exact n right eigenvalues which are complex numbers with nonnegative imaginary parts.

These eigenvalues are said to be the standard eigenvalues of A.

Lemma 2.4[4] Let $A = (a_{ij})_{n\times n} \in H^{n\times n}$. Then there exists a unitary matrix U such that U^*AU is an upper triangular matrix with diagonal entries $h_1 + k_1 i, \cdots, h_n + k_n i$, where the $h_t + k_t i$'s are the standard eigenvalues of A, i.e., $k_t \geq 0, t = 1, 2, \cdots, n$.

We denote the set of the standard eigenvalues of A by $\lambda(A)$. The upper triangular matrix is said to be the Schur upper triangular matrix of A.

Lemma 2.5[7] Let$A = (a_{ij})_{n\times n} \in H^{n\times n}$. Suppose that the standard eigenvalues of A are $\lambda_1,\ \lambda_2\ , \cdots,\ \lambda_n$. Then

$$\sum_{i=1}^{n} |\lambda_i|^2 \leq \sqrt{\|A\|_F^4 - \|AA^*\|_F^2 + \|A^2\|_F^2}.$$

Lemma 2.6[4] If $A = (a_{ij})_{n\times n} \in H^{n\times n}$ is in triangular form, then every diagonal entries is a right eigenvalue of A.

Definition 2.7[8] A quaternion matrix A is said to be a central closed matrix if there exists a invertible matrix P such that

$$P^{-1}AP = diag\left(\lambda_1, \lambda_2, \cdots, \lambda_n\right),$$

where λ_i's are real numbers.

By Lemma 2.3 and Lemma 2.6, we see that λ_i's are standard eigenvalues of A.

Recall that the trace of a square complex matrix is the sum of main diagonal entries of the matrix, which is equal to the sum of the eigenvalues of the matrix. This is not always true in quaternion case.

Example 2.8 Let

$$A = \begin{pmatrix} 1 & i \\ j & k \end{pmatrix}.$$

Then the trace of A is $1 + k$. The standard eigenvalues of A are $\lambda_1 = \frac{1+\sqrt{3}}{2} + \frac{-1+\sqrt{3}}{2}i$ and $\lambda_2 = \frac{1-\sqrt{3}}{2} + \frac{1+\sqrt{3}}{2}i$. Obviously, $Tr\left(A\right) \neq \lambda_1 + \lambda_2$. Fortunately we can show that the trace of central closed matrix is equal to the sum of the standard eigenvalues of the matrix.

Lemma 2.9[8] Let A be an $n \times n$ central closed matrix and suppose that the standard eigenvalues of A are $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n$. Then

$$Tr\left(A\right) = \sum_{i=1}^{n} \lambda_i.$$

It turns out that if A and B are $n \times n$ complex matrices, then $Tr\left(AB\right) = Tr\left(BA\right)$.

However, quaternion matrices are different from complex matrices.

Example 2.10 Let

$A = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ and $B = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$. Then $Tr\left(AB\right) \neq Tr\left(BA\right)$.

The following is the generalization of the result $Tr\,(AB) = Tr\,(BA)$ of complex matrices to quaternion matrices, which will be used in the distribution for the standard eigenvalues of quaternion matrices.

Lemma 2.11 Let $A = (a_{ij})_{n\times n} \in H^{n\times n}$ and $B = (b_{ij})_{n\times n} \in H^{n\times n}$. Then

$$Re\,(Tr\,(AB)) = Re\,(Tr\,(BA))\,.$$

For quaternion matrices, we know that left row rank is equal to right column rank. Both of them are equal to $rank\,(A)$. It is easy to show that if an $n \times n$ central closed matrix A has$s$non-zero standard eigenvalues, then $rank\,(A) \geq s$.

Lemma 2.12 Let A be an $n \times n$ central closed matrix and suppose that the standard eigenvalues of A are $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n$ and $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_s \neq 0$. Then

$$|Tr\,(A)|^2 \leq rank\,(A) \cdot \sqrt{\|A\|_F^4 - \|AA^*\|_F^2 + \|A^2\|_F^2}.$$

Proof. By Lemma 2.9, we have

$$|Tr\,(A)|^2 = \left|\sum_{i=1}^{s} \lambda_i\right|^2.$$

Using Lemma 2.5, it is easy to see the following

$$\left|\sum_{i=1}^{s} \lambda_i\right|^2 \leq s\cdot\sum_{i=1}^{s} |\lambda_i|^2 \leq rank\,(A)\cdot\sum_{i=1}^{s} |\lambda_i|^2 \leq rank\,(A)\cdot\sqrt{\|A\|_F^4 - \|AA^*\|_F^2 + \|A^2\|_F^2}.$$

So the lemma is proved.

Lemma 2.13 Let A be an $n \times n$ central closed matrix and λ be a standard eigenvalue of A. Suppose that $F\,(A) = \|AA^*\|_F^2 - \|A^2\|_F^2$, then

$$F\,(A) = F\,(\lambda I - A)\,.$$

Proof. It is easy to see that

$F\,(A) \in R, \lambda \in R$ and $F\,(\lambda I - A) \in R$.

Since

$$\begin{aligned} F\,(\lambda I - A) &= \|(\lambda I - A)\,(\lambda I - A)^*\|_F^2 - \left\|(\lambda I - A)^2\right\|_F^2 \\ &= Tr\,((\lambda I - A)\,(\lambda I - A)^*\,(\lambda I - A)\,(\lambda I - A)^*) \\ &-Tr\,((\lambda I - A)^*\,(\lambda I - A)^*\,(\lambda I - A)\,(\lambda I - A))\,, \end{aligned}$$

we have

$$F\,(\lambda I - A) = F\,(A) + Tr\,(2\lambda\,(A^*AA + A^*A^*A) - \lambda\,(A^*AA^* + AAA^* + AA^*A^* + AA^*A))\,.$$

By Lemma 2.11, we have

$$Re\,(F\,(\lambda I - A)) = Re\,(F\,(A))\,,$$

that is

$$F\,(A) = F\,(\lambda I - A)\,.$$

So the lemma is proved.

3 Main Results

We now focus on the location of the standard eigenvalues of quaternion matrices.

Theorem 3.1 Let A be an $n \times n$ central closed matrix. Then all the standard eigenvalues λ of A are located in the following Geršgorin ball:

$$G(A) = \left\{ z \in H : \left| z - \frac{Tr(A)}{n} \right| \leq R(A) \right\},$$

where

$$R(A) = \sqrt{\frac{n-1}{2n-1}} \cdot \sqrt{\frac{n-1}{n}\eta + \sqrt{\eta^2 - \frac{2n-1}{n^2}F(A)}},$$

$$\eta = \left(\|A\|_F^2 - \frac{|Tr(A)|^2}{n} \right), \quad F(A) = \|AA^*\|_F^2 - \left\|A^2\right\|_F^2.$$

Proof. Let $B = \lambda I - A$. Then

$$rank(B) = rank(\lambda I - A) \leq n - 1.$$

By Lemma 2.12, we have

$$|Tr(B)|^2 \leq rank(B) \cdot \sqrt{\|B\|_F^4 - F(B)} \leq (n-1)\sqrt{\|B\|_F^4 - F(B)}.$$

Using Lemma 2.13, we have

$$F(B) = F(A).$$

Thus

$$|Tr(\lambda I - A)|^2 \leq (n-1)\sqrt{\|\lambda I - A\|_F^4 - F(A)}. \tag{1}$$

We compute

$$\begin{aligned} |Tr(\lambda I - A)|^2 &= Tr(\lambda I - A) \cdot \overline{Tr(\lambda I - A)} \\ &= n^2\lambda^2 - \lambda n Tr\left(\overline{A}\right) - \lambda n Tr(A) + |Tr(A)|^2, \\ \|\lambda I - A\|_F^4 &= [Tr((\lambda I - A) \cdot (\lambda I - A)^*)]^2 \\ &= \left(n\lambda^2 - \lambda Tr(A^*) - \lambda Tr(A) + Tr(AA^*)\right)^2 \\ &= \left(n\lambda^2 - \lambda Tr(A^*) - \lambda Tr(A) + \|A\|_F^2\right)^2. \end{aligned}$$

Let

$$\omega = n\lambda^2 - \lambda Tr\left(\overline{A}\right) - \lambda Tr(A).$$

Then

$$|Tr(\lambda I - A)|^2 = n\omega + |Tr(A)|^2, \tag{2}$$

$$\|\lambda I - A\|_F^4 = \left(\omega + \|A\|_F^2\right)^2. \tag{3}$$

So, taking the four fundamental operations of arithmetic of (2) and (3) permits us to eliminate the unknown ω and obtain

$$\|\lambda I - A\|_F^4 = \left(\frac{1}{n}\left(|Tr(\lambda I - A)|^2 - |Tr(A)|^2\right) + \|A\|_F^2\right)^2.$$

Let $x = \left|\lambda - \frac{Tr(A)}{n}\right|^2$and $\eta = \left(\|A\|_F^2 - \frac{|Tr(A)|^2}{n}\right)$. Thus

$$\|\lambda I - A\|_F^4 = (nx + \eta)^2$$

and therefore

$$n^2 x \leq (n-1)\sqrt{(nx+\eta)^2 - F(A)}.$$

The discriminant of $n^2 x = (n-1)\sqrt{(nx+\eta)^2 - F(A)}$ is

$$\Delta = 4n^2(n-1)^2\left(n^2\eta^2 - (2n-1)F(A)\right) \geq 0.$$

Then, we obtain

$$x = \left|\lambda - \frac{Tr(A)}{n}\right|^2 \leq \frac{n-1}{2n-1} \cdot \left(\frac{n-1}{n}\eta + \sqrt{\eta^2 - \frac{2n-1}{n^2}F(A)}\right),$$

i.e.

$$\left|\lambda - \frac{Tr(A)}{n}\right| \leq \sqrt{\frac{n-1}{2n-1}} \cdot \sqrt{\left(\frac{n-1}{n}\eta + \sqrt{\eta^2 - \frac{2n-1}{n^2}F(A)}\right)}.$$

So

$$\lambda(A) \subset G(A).$$

Then all the standard eigenvalues λ of A are located in one Geršgorin ball.

Definition 3.2 Let$A = (a_{ij})_{n\times n} \in H^{n\times n}$ be given. If $A^* = A$, then A is a self-conjugate matrix over quaternion division algebra. The set of self-conjugate quaternion matrices remarked by $H(n,*)$.

Theorem 3.3 Let $A \in H(n,*)$. Then all the standard eigenvalues λ of A are located in the following Geršgorin ball

$$G(A) = \left\{z \in H : \left|z - \frac{Tr(A)}{n}\right| \leq R(A)\right\},$$

where $R(A) = \sqrt{\frac{n-1}{n}\eta}$ and $\eta = \left(\|A\|_F^2 - \frac{|Tr(A)|^2}{n}\right)$.

Proof. Since $A \in H(n,*)$, we have

$$F(A) = \|AA^*\|_F^2 - \left\|A^2\right\|_F^2 = 0.$$

Thus the theorem holds.

4 Numerical examples

Example 4.1 Let

$$A = \begin{pmatrix} 0 & j \\ 0 & 1 \end{pmatrix} \text{ and } P = \begin{pmatrix} -1 & 1 \\ j & 0 \end{pmatrix}.$$

Then
$P^{-1} = \begin{pmatrix} 0 & -j \\ 1 & -j \end{pmatrix}$, $P^{-1}AP = \begin{pmatrix} 0 & -j \\ 1 & -j \end{pmatrix} \begin{pmatrix} 0 & j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ j & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
So A is a central closed matrix. By Theorem 3.1, we have that all the standard eigenvalues of A lie in Geršgorin ball around $o = 0.5$ of radius $R \approx 0.73$. We calculate that the standard eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 0$. It is easy to see that the estimation isn't gross.
Example 4.2 Let

$$A = \begin{pmatrix} 2 & 3 & 2 \\ 10 & 3 & 4 \\ 3 & 6 & 1 \end{pmatrix} \text{ and } P = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 2 & 4 \\ -3 & -5 & 3 \end{pmatrix}.$$

Then

$$P^{-1}AP = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 11 \end{pmatrix}.$$

So A is a central closed matrix. By Theorem 3.1, we have that all the standard eigenvalues of A lie in Geršgorin ball around $o = 2$ of radius $R \approx 10.56$. We calculate that the standard eigenvalues of A are $\lambda_1 = -3$,$\lambda_2 = -2$, and $\lambda_3 = 11$. It is not hard to see that the estimation is quite accurate.
Example 4.3 Let

$$A = \begin{pmatrix} 1 & -i & -j & k \\ i & 1 & -2k & j \\ j & 2k & 7 & -i \\ -k & -j & i & 1 \end{pmatrix}.$$

Then A is a self-conjugate matrix over quaternion division algebra. By Theorem 3.3, we have that all the standard eigenvalues of A lie in Geršgorin ball around $o = 2.5$ of radius $R \approx 5.81$. We calculate that the standard eigenvalues of A are $\lambda_1 = 1$,$\lambda_2 = -1$, $\lambda_3 = 2$, and $\lambda_4 = 8$. It is not hard to see that the estimation is very sharp.

5 Conclusion

Nowadays quaternions are not only part of contemporary mathematics (algebra, analysis, geometry, and computation), but they are also widely and heavily used in programming video games and controlling spacecrafts. In this paper, we present some results for the standard eigenvalues of quaternion matrices. These results may be useful to further study other algebra problems over quaternion division algebra.

Let A be a central closed matrix. Since $P^{-1}AP$ has the same standard eigenvalues as A whenever P is invertible. We can apply the Theorem 3.1 to $P^{-1}AP$, perhaps for some choice of P the bounds obtained may be sharper. Now, we have three questions as follows.

Question 1 What kind of P satisfy $R\left(P^{-1}AP\right) < R\left(A\right)$?

Question 2 What is the minimal radius of $G\left(A\right)$?

Question 3 When we find the minimal radius of $G\left(A\right)$, is $G\left(A\right)$ the minimal Gerschgorin ball which contains all the standard eigenvalues of A? How can we find the minimal Gerschgorin ball which contains all the standard eigenvalues of A?

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