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Category of Chain Complexes of Soft Modules

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Abstract. Molodtsov initiated the concept of soft sets in [4]. Maji et al. defined some operations on soft sets in [10]. Aktaş et al. generalized soft sets by defining the concept of soft groups in [7]. After then, Qiu-Mei Sun et al. gave soft modules in [12]. In this paper, methods of homology algebra are applied to the category of soft modules. For this reason, chain complexes of soft modules and their soft homology modules are defined. For defined soft homology modules, the axioms of homology theory provide are shown.

Keywords: soft set, soft modules, soft homomorphism, chain complexes of soft modules

1. INTRODUCTION

Many practical problems in economics, engineering, environment, social science, medical science etc. cannot be dealt with by classical methods, because classical methods have inherent difficulties. The reason for these difficulties may be due to the inadequacy of the theories of parameterization tools. Molodtsov [4] initiated the concept of soft set theory as a new mathematical tool for dealing with uncertainties. Then P. K. Maji [9] initiated the concept of fuzzy soft theory. Soft sets and fuzzy soft sets theory has a rich potential for applications in several directions, few of which had been shown by some authors [4][11][2]. The algebraic structure of set theories dealing with uncertainties has also been studied by some authors. After Molodtsov's work, some different applications of soft sets were studied in [11]. Aktaş and Çağman [7] defined soft groups and compared soft sets with fuzzy sets and rough sets.F. Feng et al. [5] gave soft semirings and U. Acar et al. [13] introduced initial concepts of soft rings. Qiu-Mei Sun et al.[12] defined soft modules and investigated their basic properties. Structures of fuzzy soft group and fuzzy soft module are obtained by some authors [3][8][1], and some properties of these concepts are investigated.

In this paper, the methods of homology algebra are applied to the category of soft modules. For this reason, chain complexes of soft modules and their soft homology modules are defined. For defined soft homology modules, the axioms of homology theory provide are shown.

2. PRELİMİNARY

In this section, we recall some basic concepts of set theory. Throughout this subsection U refers to on initial universe E is a set of parameters, $P(U)$ is the power set of U, and $A \subseteq E$.

Definition 1. [4] *A pair (F, A) is called a soft set over U where F is a mapping given by $F : A \to P(U)$. In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε-elements of the soft set (F, A), or as the set of ε-approximate elements of the soft set.*

Definition 2. [10] *For two soft sets (F, A) and (G, B) over U, (F, A) is called soft subset of (G, B) if*

1. *$A \subset B$ and*
2. *$\forall \varepsilon \in A$, $F(\varepsilon)$ and $G(\varepsilon)$ are identical approximations. This relationship is denoted by $(F, A) \tilde{\subset} (G, B)$.*

Similarly, (F, A) is called a soft superset of (G, B) if (G, B) is a soft subset of (F, A). This relationship is denoted by $(F, A) \tilde{\supset} (G, B)$.

Two soft sets (F, A) and (G, B) over $U$are called soft equal if (F, A) is a soft subset of (G, B), and (G, B) is a soft subset of (F, A).

Definition 3. [10] *The intersection of two soft sets (F, A) and (G, B) over U is the soft set (H, C), where $C = A \cap B$ and $\forall \varepsilon \in C$, $H(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$. This is denoted by $(F, A) \tilde{\cap} (G, B) = (H, C)$.*

Definition 4. [10] *If (F, A) and (G, B) are two soft sets, then (F, A) AND (G, B) is denoted $(F, A) \wedge (G, B)$. $(F, A) \wedge (G, B)$ is defined as $(H, A \times B)$, where $H(\alpha, \beta) = F(\alpha) \cap G(\beta)$,$\forall (\alpha, \beta) \in A \times B$.*

Definition 5. [10] *The union of two soft sets (F, A) and (G, B) over is the soft set (H, C), where $C = A \cup B$ and $\forall \varepsilon \in C$,*

$$H(\varepsilon) = \begin{cases} F(\varepsilon), & \text{if } \varepsilon \in A - B, \\ G(\varepsilon), & \text{if } \varepsilon \in B - A, \\ F(\varepsilon) \cup G(\varepsilon) & \text{if } \varepsilon \in A \cup B. \end{cases}$$

This relationship is denoted by $(F, A) \tilde{\cup} (G, B) = (H, C)$.

Throughout this subsection, let M be a left R-module, A be any nonempty set. $F : A \to P(M)$ refer to a set-valued function and the pair (F, A) is a soft set over M.

Definition 6. [12] *Let (F, A) be a soft set over M.(F, A) is said to be a soft module over M if and only if $F(x) < M$ for all $x \in A$.*

Proposition 1. [12] *Let (F, A) and (G, B) be two soft modules over M.*

1. *$(F, A) \,\widetilde{\cap}\, (G, B)$ is a soft module over M.*
2. *$(F, A) \,\widetilde{\cup}\, (G, B)$ is a soft module over M if $A \cap B = \emptyset$.*

Definition 7. [12] *Let (F, A) and (G, B) be two soft modules over M. Then $(F, A) + (G, B)$ is defined as $(H, A \times B)$, where $H(x, y) = F(x) + G(y)$ for all $(x, y) \in A \times B$.*

Proposition 2. [12] *Let (F, A) and (G, B) be two soft modules over M. Then $(F, A) + (G, B)$ is soft module over M.*

Definition 8. [12] *Let (F, A) and (G, B) be two soft modules over M and N respectively. Then $(F, A) \times (G, B) = (H, A \times B)$ is defined as $H(x, y) = F(x) \times G(y)$ for all $(x, y) \in (A \times B)$.*

Proposition 3. [12] *Let (F, A) and (G, B) be two soft modules over M and N respectively. Then $(F, A) \times (G, B)$ is soft module over $M \times N$.*

Direct product and direct sum the same when the dimension is finite, so " $\oplus$ " can instead of " $\times$ " in the above proposition.

Now, let parameter set of $\{(F_i, A_i)\}_{i \in I}$ be fixed point. We denote fixed point of A_i as a_{0i} and let $F_i(a_{0i}) = 0$. For $A = \prod_{i \in I} A_i$and $M = \bigoplus_{i \in I} M_i$, we define the mapping $F : A \to M$ by $F(a) = \bigoplus_{i \in I} F(a_i)$, for all $a = \{a_i\} \in A$. Then, (F, A) is a soft module over M[3].

Definition 9. [3]*(F, A) is said to be direct sum of $\{(F_i, A_i)\}_{i \in I}$ and denoted as $\bigoplus_{i \in I} (F_i, A_i)$.*

The mapping $\varphi_j : A_j \to \prod_{i \in I} A_i$ is defined by $\varphi_j(a_j) = \{a_i\}$ such that if $i \neq j$, then $a_i = a_{0i}$ and if $i = j$, then $a_i = a$. Also for embedding mapping $q_j : M_j \to \bigoplus_{i \in I} M_i$, $(q_j, \varphi_j) : (F_j, A_j) \to (F, A)$ is a soft homomorphism of soft modules [3].

Definition 10. [12] *Let (F, A) and (G, B) be two soft modules over M. Then (G, B) is soft module of (F, A) if*

1. *$B \subset A$ and*
2. *$G(x) < F(x)$, $\forall x \in B$*
   this denoted by $(G, B) \widetilde{<} (F, A)$.

Proposition 4. [12] *Let (F, A) and (G, B) be two soft modules over M, and (G, B) be soft submodule of (F, A). If $f : M \to N$ is homomorphism of module, then $(f(F), A)$ and $(f(G), B)$ are all soft modules over N, and $(f(G), B)$ is soft submodule of $(f(F), A)$.*

Definition 11. [12] *Let (F, A) and (G, B) be two soft modules over M and N respectively, $f : M \to N$, $g : A \to B$ be two functions. Then we say that (f, g) is a soft homomorphism if the following conditions are satisfied:*

1. *$f : M \to N$ is homomorphism of module;*
2. *$g : A \to B$ is a mapping;*
3. *$f(F(x)) = G(g(x))$, $\forall x \in A$.*

At the same time, we say (F, A) is soft homomorphic to (G, B), which denoted by $(F, A) \simeq (G, B)$.

In this definition, if f is an isomorphism from M to N and g is a one-to-one mapping from A onto *B, then we say that (F, A) is a soft isomorphism and that (F, A) is a soft isomorphic to (G, B), this is denoted by $(F, A) \cong (G, B)$.*

3. CHAIN COMPLEXES OF SOFT MODULES

Throughout this section, let M be a left R-module, A be any nonempty set.
Let (F, A) and (G, B) be two soft modules over M and N respectively, and $(f, g) : (F, A) \to (G, B)$ be a soft homomorphism of these modules.
Now in this section we introduce the kernel and image of soft homomorphism of soft modules. Let $M' = \ker f$ and

$$A' = \{x \in A |\, F(x) < \ker f\}.$$

$F' = F_{A'}$ is defined by $F' : A' \to P(M')$. Then (F', A') is a soft module over M'.

Definition 12. *(F', A') is said to be kernel of (f, g), and denoted by* $\ker(f, g)$.

Now, let $N' = Imf < N$ and

$$B' = \{x \in B |\, G(x) < Imf\}.$$

$G' = G_{B'}$is defined by $G' : B' \to P(N')$. Then (G', B') is a soft module over N' and is a soft submodule of (G, B).

Definition 13. *(G', B') is said to be image of (f, g), and denoted by $Im(f, g)$.*

Let $\{(F_n, A)\}_{n\in \mathrm{Z}}$ be soft modules over $\{M_n\}_{n\in\mathbb{Z}}$, and let for all $n \in \mathbb{Z}$,$(\partial_n, 1_A) : (F_n, A) \to (F_{n-1}, A)$ be homomorphism of soft modules.

Definition 14. *If for all $a \in A$ $\left\{(F_n(a)), \partial_n|_{F_n(a)} : F_n(a) \to F_{n-1}(a)\right\}$ is a chain complex of modules, that is, $\partial_{n-1}|_{F_{n-1}(a)} \circ \partial_n|_{F_n(a)} = 0$, then the following sequence is said to be a chain complex of soft modules.*

$$\{(F_n, A), (\partial_n, 1_A) : (F_n, A) \to (F_{n-1}, A)\}$$

Definition 15. *If the condition $Im\partial_n|_{F_n(a)} = \ker \partial_{n-1}|_{F_{n-1}(a)}$ is satisfied at the chain complex$\left\{F_n(a), \partial_n|_{F_n(a)} : F_n(a) \to F_{n-1}(a)\right\}$, then the following sequence is said to be an exact sequence of soft modules*

$$\{(F_n, A), (\partial_n, 1_A) : (F_n, A) \to (F_{n-1}, A)\}$$

.

Now, let us define morphisms of the chain complexes of soft modules.

Definition 16. *Let* $\{(F_n, A), \partial_n\}$, $\{(G_n, B), \partial'_n\}$ *be chain complexes of soft modules over* $\{M_n\}$ *and* $\{N_n\}$, *respectively and let* $\{f_n : M_n \to N_n\}_n$ *be homomorphism of modules, and* $g : A \to B$ *is a mapping of sets. If the following diagram is commutative, for all* $a \in A$,

$$\begin{array}{ccc} F_n(a) & \xrightarrow{\partial_n} & F_{n-1}(a) \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n-1}} \\ G_n(g(a)) & \underset{\partial'_n}{\longrightarrow} & G_{n-1}(g(a)) \end{array}$$

then $(\{f_n\}, g) : \{(F_n, A), \partial_n\} \to \{(G_n, B), \partial'_n\}$ *is said to be morphism of chain complexes of soft modules.*

Remark 1. *Chain complexes of soft modules and morphisms of their forms a category. This category is denoted by* $CCSM$.

Definition 17. *Let* $(\{\varphi_n\}, g), (\{\psi_n\}, g) : \{(F_n, A), \partial_n\} \to \{(G_n, B), \partial'_n\}$ *be morphisms of chain complex of soft modules, and let* $D = \{(D_n, g) : (F_n, A) \to (G_{n+1}, B)\}$ *be a family of homomorphisms of soft modules. If the equation* $\varphi_n - \psi_n = D_{n-1} \circ \partial_n + \partial'_{n+1} \circ D_n$ *is satisfied, then the family of homomorphisms of modules* $D = \{(D_n, g) : M_n \to N_{n+1}\}_{n\in\mathbb{Z}}$ *is said to be a chain homotopy morphisms,*$(\{\varphi_n\}, g)$, $(\{\psi_n\}, g)$*is said to be a chain homotopy mappings and denoted by* $(\{\varphi_n\}, g) \sim (\{\psi_n\}, g)$.

Theorem 1. *Chain homotopy relation is a equivalence relation, and is invariant according to composition.*

Proof. Primarily, we show that chain homotopy relation is a equivalence relation.
1) Let $(\varphi, g) = (\{\varphi_n\}, g) : \{(F_n, A), \partial_n\} \to \{(G_n, B), \partial'_n\}$ be an arbitrary morphism. If $D_n = 0$, then $\varphi_n - \varphi_n = 0$. That is, $(\varphi, g) \sim (\varphi, g)$.
2) Let (φ, g) with (ψ, g) be a chain homotopy. That is,

$$D_{n-1} \circ \partial_n + \partial'_{n+1} \circ D_n = \varphi_n - \psi_n.$$

If $\bar{D}_n = -D_n$, in this case

$$\begin{aligned} \bar{D}_{n-1}\partial_n + \partial'_{n+1}\bar{D}_n &= -D_{n-1}\partial_n - \partial'_{n+1}D_n \\ &= -(D_{n-1}\partial_n + \partial'_{n+1}D_n) \\ &= -(\varphi_n - \psi_n) \\ &= \psi_n - \varphi_n \end{aligned}$$

and then (φ, g) with (ψ, g) is a chain homotopy.
3) Let (φ, g) with (ψ, g) and (ψ, g) with (γ, g) be a chain homotopy. We want to show that (φ, g) with (γ, g) is a chain homotopy. If (φ, g) with (ψ, g) is a chain homotopy,$\exists D_n \Rightarrow D_{n-1}\partial_n + \partial'_{n+1}D_n = \varphi_n - \psi_n$. If (ψ, g) with (γ, g) is a chain homotopy,$\exists D'_n \Rightarrow D'_{n-1}\partial_n + \partial'_{n+1}D'_n = \psi_n - \gamma_n$.

Let define the homomorphism D''_n as $D''_n = D_n + D'_n$.

$$\begin{aligned} D''_{n-1}\partial_n + \partial'_{n+1}D''_n &= (D_{n-1} + D'_{n-1})\partial_n + \partial'_{n+1}(D_n + D'_n) \\ &= D_{n-1}\partial_n + D'_{n-1}\partial_n + \partial'_{n+1}D_n + \partial'_{n+1}D'_n \\ &= D_{n-1}\partial_n + \partial'_{n+1}D_n + D'_{n-1}\partial_n + \partial'_{n+1}D'_n \\ &= (\varphi_n - \psi_n) + (\psi_n - \gamma_n) \\ &= \varphi_n - \gamma_n \end{aligned}$$

Now, we show the invariance of composition.

$$(\{\varphi_{0n}\}, g) \sim (\{\psi_{0n}\}, g) : [\{(F_n, A), \partial_n\} \to \{(G_n, B), \partial'_n\}] \Rightarrow D_{n-1}\partial_n + \partial'_{n+1}D_n = \varphi_{0n} - \psi_{0n}$$

$$(\{\varphi_{1n}\}, h) \sim (\{\psi_{1n}\}, h) : [\{(G_n, B), \partial'_n\} \to \{(P_n, C), \partial''_n\}] \Rightarrow D'_{n-1}\partial_n + \partial''_{n+1}D'_n = \varphi_{1n} - \psi_{1n}$$

For $(\{\varphi_{1n}\}, h) \circ (\{\varphi_{0n}\}, g), (\{\psi_{1n}\}, h) \circ (\{\psi_{0n}\}, g) : [\{(F_n, A), \partial_n\} \to \{(P_n, C), \partial''_n\}]$ to be a chain homotopy we have to define a homomorphism in following the form

$$(D''_n, \omega) : [\{(F_n, A), \partial_n\} \to \{(H_n, C), \partial''_n\}].$$

$$\begin{aligned} D'_{n-1}(\varphi_{0n-1}, g)\partial_n + \partial''_{n+1}D'_n(\varphi_{0n}, g) &= D'_{n-1}(\partial'_n(\varphi_{0n}, g)) + \partial''_{n+1}D'_n(\varphi_{0n}, g) \\ &= (D'_{n-1}\partial'_n + \partial''_{n+1}D'_n)(\varphi_{0n}, g) \\ &= (\varphi_{1n}, h)(\varphi_{0n}, g) - (\psi_{1n}, h)(\varphi_{0n}, g) \end{aligned}$$

Now, we show that $(\{\psi_{1n}\}, h) \circ (\{\varphi_{0n}\}, g)$with $(\{\psi_{1n}\}, h) \circ (\{\psi_{0n}\}, g)$are chain homotopy. We are look at $(\psi_{1n+1}, h) \circ D_n : F_n(a) \to H_{n+1}(h(g(a)))$.

$$\begin{aligned} (\psi_{1n}, h)D_{n-1}\partial_n + \partial''_{n+1}(\psi_{1n+1}, h)D_n &= (\psi_{1n}, h)D_{n-1}\partial_n + (\psi_{1n}, h)\partial'_{n+1}D_n \\ &= (D_{n-1}\partial_n + \partial'_{n+1}D_n)(\psi_{1n}, h) \\ &= ((\varphi_{0n}, g) - (\psi_{0n}, g))(\psi_{1n}, h) \\ &= (\varphi_{0n}, g)(\psi_{1n}, h) - (\psi_{0n}, g)(\psi_{1n}, h) \end{aligned}$$

Then $(\varphi_{0n}, g) \circ (\psi_{1n}, h)$ with $(\psi_{1n}, g) \circ (\psi_{0n}, h)$ is a chain homotopy. Hence, from the two equalities, $(\{\varphi_{1n}\}, h) \circ (\{\varphi_{0n}\}, g)$ with $(\{\psi_{1n}\}, h) \circ (\{\psi_{0n}\}, g)$ is a chain homotopy. □

Let $F = \{(F_n, A), \partial_n\}$ be a chain complex of soft modules over $\{M_n\}$. We obtain the homology module

$$H_n(F, a) = \ker \partial_n|_{Im\partial_{n+1}}$$

for the chain complex $\{F_n(a), \partial_n : F_n(a) \to F_{n-1}(a)\}$ and $\forall a \in A$. Thus, for $\forall a \in A$ the module $H_n(F, a)$ is a quotient module in $M_n$module. If there exist an one-to-one and covered connection with every submodule of quotient module of M and submodule of M_n we can think the module $H_n(F, a)$ as a submodule of M_n.
Thus, $H_n(F, -) : A \to P(M_n)$ is a soft module.

Definition 18. *Soft module $(H_n(F, -), A)$ is said to be n-dimensional homology soft module of chain complexes of soft modules $\{(F_n, A), \partial_n\}$.*

Now, we show that homology soft module is a functor.
Let $(\varphi = \{\varphi_n\}, g) : \{(F_n, A), \partial_n\} \to \{(G_n, B), \partial'_n\}$ be morphisms of chain complexes of soft modules. Since $\{\varphi_n : F_n(a) \to G_n(g(a))\}$ is morphism of chain complexes for all $a \in A$, mapping $\varphi_{n_*} : H_n(F, a) \to H_n(G, g(a))$, defined by $\varphi_{n_*}[x] = [\varphi_n(x)]$ for all $[x] \in H_n(F, a)$, is a homomorphism of modules, and the following diagram is commutative

$$\begin{array}{ccc} A & \overset{H_n(F,-)}{\longrightarrow} & P(M_n) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f_{n*}} \\ B & \underset{H_n(G,-)}{\longrightarrow} & P(N_n) \end{array} .$$

Then $(\varphi_{n_*}, g) : (H_n(F, -), A) \to (H_n(G, -), B)$is homomorphism of soft modules.

Theorem 2. *The corresponding $F \mapsto H_n(F, -), (\{\varphi_n\}, g) \mapsto (\varphi_{n_*})$is a covariant functor from the category $CCSM$ to the category SM.*

Theorem 3. *Homology functor of chain complexes of soft modules is invariant according to chain homotopy. That is, if*

$$\{\varphi_n\} \sim \{\psi_n\} : \{(F_n, A), \partial_n\} \to \{(G_n, A), \partial'_n\}$$

then $\varphi_{n_*} = \psi_{n_*} = (H_n(F, -), A) \to (H_n(G, -), A)$.

Proof. Since $\{\varphi_n\}$ with $\{\psi_n\}$ is a chain homotopy for all $a \in A$, then

$$\exists D_n : (F_n, A) \to (G_{n+1}, B)$$

such that the equation

$$D_{n-1}\partial_n + \partial'_{n+1} D_n = \varphi_n - \psi_n \tag{3.1}$$

is satisfied. Now, we show that $\varphi_{n_*} = \psi_{n_*} = (H_n(F, -), A) \to (H_n(G, -), A)$ is satisfied. For $\forall a \in A$ and $\forall [z] = z + Im\partial_{n+1} \in H_n(F, a)$ we want to show that

$$\varphi_{n_*}(z + Im\partial_{n+1}) = \psi_{n_*}(z + Im\partial_{n+1}).$$

That is, $\varphi_{n_*}(z + Im\partial_{n+1}) = \varphi_n(z) + Im\partial'_{n+1}$. We show that

$$\psi_{n_*}(z + Im\partial_{n+1}) = \psi_n(z) + Im\partial'_{n+1}.$$

Since $z \in \ker \partial_n$, and from the equation 3.1,

$$D_{n-1}\partial_n(z) + \partial'_{n+1} D_n(z) = \partial'_{n+1}(D_n(z)) = \varphi_n(z) - \psi_n(z)$$

$$\begin{aligned} &\Rightarrow a = \varphi_n(z) - \psi_n(z) \quad \exists b \in D_n(z) \quad \partial'_{n+1}(b) = a \\ &\Rightarrow a \in Im\partial'_{n+1} \\ &\Rightarrow \varphi_n(z) - \psi_n(z) \in Im\partial'_{n+1} \end{aligned}$$

$$[\varphi_n(z)] = \varphi_n(z) + Im\partial'_{n+1} \qquad [\psi_n(z)] = \psi_n(z) + Im\partial'_{n+1}$$

□

Now, the following definition can be given.

$$C = \{(F_n, B), (\partial_n, 1_B) : (F_n, B) \to (F_{n-1}, B)\}$$
$$C' = \left\{(F'_n, A), (\partial'_n, 1_A) : (F'_n, A) \to (F'_{n-1}, A)\right\}$$
$$C'' = \left\{(F''_n, E), (\partial''_n, 1_E) : (F''_n, E) \to (F''_{n-1}, E)\right\}$$

$$(i, \mu) = \{i_n : F'_n(a) \to F_n(\mu(a))\} \quad (j, \eta) = \{j_n : F_n(\mu(a)) \to F''_n(\eta(\mu(a)))\}$$

Definition 19. *Let C, C', C'' be chain complexes of soft modules, and let $(i, \mu) : C' \to C, (j, \eta) : C \to C''$ be morphisms of chain complexes. If the following sequence of modules is exact for $a \in A$,*

$$0 \xrightarrow{0} F'_n(a) \xrightarrow{i_n} F_n(\mu(a)) \xrightarrow{j_n} F''_n(\eta(\mu(a))) \xrightarrow{0} 0$$

then the following sequence of chain complexes of soft modules is said to be short exact sequence

$$0 \xrightarrow{0} C' \xrightarrow{(i,\mu)} C \xrightarrow{(j,\eta)} C'' \xrightarrow{0} 0\ .$$

Let the following C, C' be chain complexes of soft modules.

$$C = \{(F_n, A), (\partial_n, 1_A) : (F_n, A) \to (F_{n-1}, A)\}$$
$$C' = \left\{(F'_n, A), (\partial'_n, 1_A) : (F'_n, A) \to (F'_{n-1}, A)\right\}$$

Definition 20. *If $F'_n(a)$ is a submodule of $F_n(a)\,(F'_n(a) < F_n(a))$for all $a \in A$, then C' is said to be submodule of C.*

Definition 21. *If $C' < C$, and if the mapping $F_n|_{F'_n} : A \to P(M_n)$ is defined as $\left(F_n|_{F'_n}\right)(a) = F_n(a)|_{F'_n(a)}$, then $\left\{\left(F_n|_{F'_n}, A\right)\right\}$ is a chain complex of soft modules. This chain complex is called as ${}^{C}/_{C'}$ quotient complex.*

Obviously, if $C' < C$, then the following sequence of chain complexes of soft modules is exact

$$0 \to C' \to C \to {}^{C}/_{C'} \to 0\ .$$

Definition 22. *Soft homology module of the complex $C\ /C'$ is said to be soft homology module of the pair (C, C') and defined by $H_n(C, C')$.*

Theorem 4. *For every pair of chain complexes of soft modules (C, C') there exists homomorphism of soft modules $\partial_{n*} : H_n(C, C') \to H_{n-1}(C')$ such that*

1. *The following sequence of soft homology modules is exact.*

$$(3.2) \qquad - - - \leftarrow H_{n-1}(C') \xleftarrow{\partial_{n*}} H_n(C, C') \longleftarrow H_n(C) \longleftarrow H_n(C') \leftarrow - - -$$

2. *If* $(\varphi, g) : (C, C') \to (K, K')$ *is a morphism of chain complexes of soft modules, then the following diagram is commutative.*

$$\begin{array}{ccc} H_n(C,C') & \xrightarrow{\partial_{n*}} & H_{n-1}(C') \\ {\scriptstyle H_n(\varphi,g)}\downarrow & & \downarrow{\scriptstyle H_{n-1}(\varphi,g)} \\ H_n(K,K') & \xrightarrow[\partial'_{n*}]{} & H_{n-1}(K') \end{array}$$

Proof. We obtain the following short exact sequence of chain complexes of modules for all $a \in A$.

$$0 \to F'(a) \to F(a) \to {}^{F(a)}\big/_{F'(a)} \to 0 \tag{3.3}$$

In this case, the homomorphism $\partial_{n*}(a) : H_n(F(a), F'(a)) \to H_{n-1}(F'(a))$ can be defined, and

$$(\partial_{n*}, 1_A) : \{(H_n(F, F'), A) \to (H_{n-1}(F'), A)\}$$

is a homomorphism of soft modules.The following exact sequence of homology modules is obtained from the sequence (3.3).

$$- - - \leftarrow H_{n-1}(F'(a)) \leftarrow H_n(F(a), F'(a)) \leftarrow H_n(F(a)) \leftarrow H_n(F'(a)) \leftarrow - - - \tag{3.4}$$

Since the sequence (3.4) is exact for all $a \in A$, the sequence of soft homology modules (3.2) is exact. If $(\varphi, g) : (C, C') \to (K, K')$is a morphism of chain complexes of soft modules, and since $\varphi = \{\varphi_n\} : \{F_n(a)\} \to \{G_n(g(a))\}$ for all $a \in A$ is a morphism of chain complex of modules, the following diagram is commutative.

$$\begin{array}{ccc} H_n(F'(a)) & \xleftarrow{\partial_{n+1*}} & H_{n+1}(F(a), F'(a)) \\ {\scriptstyle H_n(\varphi,g)}\downarrow & & \downarrow{\scriptstyle H_{n+1}(\varphi,g)} \\ H_n(G'(a)) & \xleftarrow[\partial'_{n+1*}]{} & H_{n+1}(G(g(a)), G'(g(a))) \end{array}$$

From here, the following diagram is commutative for soft homology modules

$$\begin{array}{ccc} H_n(F') & \longleftarrow & H_{n+1}(F, F') \\ \downarrow & & \downarrow \\ H_n(G') & \longleftarrow & H_{n+1}(G, G') \end{array} .$$

□

Let $(F, A), (G, B)$ and $(F', A'), (G', B')$ are soft modules over M and N respectively, and let

$$\begin{array}{l} (f, g_A) : (F, A) \to (F', A') \\ (h, g_B) : (G, B) \to (G', B') \end{array}$$

be a homomorphism of soft modules. For the union of soft modules $(F, A)\widetilde{\cup}(G, B)$,$(F', A')\widetilde{\cup}$ (G', B') we have $C = A \cup B, C' = A' \cup B', A \cup B = \emptyset$,

$$H(x) = \begin{cases} F(x) & for\ x \in A - B \\ G(x) & for\, x \in B - A \end{cases} \qquad H'(x) = \begin{cases} F'(x) & for\ x \in A' - B' \\ G'(x) & for\, x \in B' - A' \end{cases} .$$

For all $c \in C$, the mapping $g : C \to C'$ and $\varphi_c : M \to N$ are defined by

$$g(x) = \begin{cases} g_A(x) & for\ x \in A - B \\ g_B(x) & for\ x \in B - A \end{cases} \qquad \varphi_c(x) = \begin{cases} f(x) & for\ c \in A - B \\ h(x) & for\ c \in B - A \end{cases}.$$

Since the following diagram is commutative,

$$\begin{array}{ccc} C & \xrightarrow{H} & P(M) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \varphi} \\ C' & \xrightarrow[H']{} & P(N) \end{array}$$

$(\varphi_c, g) : (H, C) \to (H', C')$ is a homomorphism of soft modules.

Theorem 5. $\tilde{\cup} : SM \times SM \to SM$ *is a functor.*

Let $(F, A), (G, B)$ and $(F', A'), (G', B')$ are soft modules over M and N respectively, and let

$$(f, g_A) : (F, A) \to (F', A')$$
$$(h, g_B) : (G, B) \to (G', B')$$

be a homomorphism of soft modules. For the intersection of soft modules $(F, A) \tilde{\cap} (G, B)$, $(F', A') \tilde{\cap} (G', B')$ we have $C = A \cap B$, $C' = A' \cap B'$, $H(x) = F(x) \cap G(x)$ and $H'(x) = F'(x) \cap G'(x)$.
Under the conditions $g_A|_{A\cap B} = g_B|_{A\cap B}$ and $f|_{F(c)\cap G(c)} = h|_{F(c)\cap G(c)}$, and for all $c \in C$, the mappings $g : C \to C'$ and $\varphi_c : M \to N$ are defined by

$$g(x) = \begin{cases} g_A(x) & for\ x \in A - B \\ g_B(x) & for\ x \in B - A \\ g_A(x) = g_B(x) & for\ x \in A \cap B \end{cases} \qquad \varphi_c(x) = \begin{cases} f(x) & for\ c \in A - B \\ h(x) & for\ c \in B - A \\ f(x) = h(x) & for\ c \in A \cap B \end{cases}.$$

Since the following diagram is commutative

$$\begin{array}{ccc} C & \xrightarrow{H} & P(M) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \varphi} \\ C' & \xrightarrow[H']{} & P(N) \end{array}$$

$(\varphi_c, g) : (H, C) \to (H', C')$ is a homomorphism of soft modules.

Theorem 6. *Under the conditions* $g_A|_{A\cap B} = g_B|_{A\cap B}$ *and* $f|_{F(c)\cap G(c)} = h|_{F(c)\cap G(c)}$, $\tilde{\cap}$: $SM \times SM \to SM$ *is a functor.*

Let $(F, A), (G, B)$ and $(F', A'), (G', B')$ are soft modules over M and N respectively, and let

$$(f, g_A) : (F, A) \to (F', A')$$
$$(h, g_B) : (G, B) \to (G', B')$$

be a homomorphism of soft modules. For the operation "AND" of soft modules $(F,A)\wedge(G,B)$,$(F',A')\wedge(G',B')$ we have $C=A\times B$,$C'=A'\times B'$, $H(x,y)=F(x)\cap G(y)$ (for all $(x,y)\in A\times B=C$) and $H'(x)=F'(x)\cap G'(x)$(for all $(x,y)\in A'\times B'=C'$).
Under the condition $f|_{F(x)\cap G(y)}=h|_{F(x)\cap G(y)}$, and for all $c\in C$, the mappings$g:C\to C'$ and $\varphi_c:M\to N$ are defined by

$$g=g_A\times g_B:A\times B\to A'\times B' \qquad \varphi_c(x)=\begin{cases} f(x) & for\, c\in A-B\\ h(x) & for\, c\in B-A\\ f(x)=h(x) & for\, c\in A\cap B\end{cases}.$$

Since the following diagram is commutative

$$\begin{array}{ccc} C & \xrightarrow{H} & P(M)\\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \varphi}\\ C' & \xrightarrow[H']{} & P(N)\end{array}$$

$(\varphi_c,g):(H,C)\to(H',C')$ is a homomorphism of soft modules.
Let $C=\{(F_n,A),\partial_n\}, C'=\{(F'_n,A'),\partial'_n\}$ be a chain complex of soft modules.

Proposition 5. *Also* $\left\{(F_n,A)\,\tilde{\cup}\,(F'_n,A'),\overline{\partial}_n\right\}$ *is a chain complex of soft modules. Here, since* $A\cap A'=\emptyset$ *we have*

$$\overline{\partial_n}=\begin{cases}\partial_n & for\, a\in A-A'\\ \partial'_n & for\, a\in A'-A\end{cases}.$$

Proposition 6. *Also* $\left\{(F_n,A)\,\tilde{\cap}\,(F'_n,A'),\overline{\partial}_n\right\}$ *is a chain complex of soft modules. Here, we have*

$$\overline{\partial_n}=\begin{cases}\partial_n & for\, a\in A-A'\\ \partial'_n & for\, a\in A'-A\\ \partial_n=\partial'_n & for\, a\in A\bigcap A'\end{cases}.$$

Theorem 7. *The following qualities can be given about the soft homology modules.*

$$H_n(F\,\tilde{\cup}\,F',c)=\begin{cases} H_n(F,c) & c\in A\\ H_n(F',c) & c\in A'\end{cases}$$

$$H_n(F\,\tilde{\cap}\,F',c)=H_n(F,c)\cap H_n(F',c)$$

4. CONCLUSION

This paper summarized the basic concept of soft sets and soft modules. By using these concepts, we studied the algebraic properties of soft sets in module structure. This work focused on soft module, chain complexes of soft modules.

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