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A Characterization of Minimal Surfaces in the Lorentz Group L^3

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Abstract

In this paper we establish the equation for the Gaussian Curvature of a minimal surface in the Lorentz Group $\mathbb{L}^3$. Using the Gauss equation we prove that minimal surfaces in $\mathbb{L}^3$ with constant contact angle have non-positive Gaussian curvature. Also, we provide a congruence theorem for minimal surfaces immersed in the Lorentz space $\mathbb{L}^3$.

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1 Introduction

In [10] we introduced the notion of contact angle, which can be considered as a new geometric invariant useful for investigating the geometry of immersed surfaces in S^3. Geometrically, the contact angle (β) is the complementary angle between the contact distribution and the tangent space of the surface. Also in [10], we derived formulae for the Gaussian curvature and the Laplacian of an immersed minimal surface in S^3, and we gave a characterization of the Clifford Torus as the only minimal surface in S^3 with constant contact angle. Besides, interesting characterizations of the Clifford torus in spheres are given in [14] and [15]. Also, examples of minimal surfaces in the Heisenberg group was studied in [1], [2] and [5] . Moreover in [11], we construct a family of minimal tori in S^5 with constant contact angle and constant holomorphic angle. These tori are parametrized by the following circle equation

$$a^2 + \left(b - \frac{\cos\beta}{1+\sin^2\beta}\right)^2 = 2\frac{\sin^4\beta}{(1+\sin^2\beta)^2}, \tag{1}$$

In particular, when $a = 0$, we recover the examples found by Kenmotsu, in [6], [7] and [8]. These examples are defined for $0 < \beta < \frac{\pi}{2}$. Also, when $b = 0$, we find a new family of minimal tori in S^5, and these tori are defined for $\frac{\pi}{4} < \beta < \frac{\pi}{2}$. For $\beta = \frac{\pi}{2}$, we give an alternative proof of this classification of a Theorem from Blair in [3], and Yamaguchi, Kon and Miyahara in [9] for Legendrian minimal surfaces in S^5 with constant Gaussian curvature.
Also in [12] we provide a congruence theorem for minimal surfaces in S^5 with constant contact angle using Gauss-Codazzi-Ricci equations. More precisely, we prove that Gauss-Codazzi-Ricci equations for minimal surfaces in S^5 with constant contact angle satisfy an equation for the Laplacian of the holomorphic angle. Also, we will give a characterization of flat minimal surfaces in S^5 with constant contact angle.
The scope of this note is to use a geometric invariant in order to study immersed surfaces in the three dimensional Lorentz group $\mathbb{L}^3$. This invariant (the contact angle (β)) is the complementary angle between the contact distribution and the tangent space of the surface.
We show that the Gaussian curvature K of a minimal surface in $\mathbb{L}^3$ with contact angle β is given by:

$$K = 1 - |\nabla\beta + (\cosh^2(\beta) + \sinh^2(\beta))e_1|^2$$

Using the equation of Gauss , we have proved the following theorem:

Theorem 1. *The Gaussian Curvature for minimal surfaces in $\mathbb{L}^3$ with constant contact angle is non-positive.*

Therefore, we have the following observation:

Remark 1. *There are no minimal surfaces in $\mathbb{L}^3$ with $K > 0$ and constant contact angle.*

More in general, we have the following congruence result:

Theorem 2. *Consider S a Riemannian surface, $\boldsymbol{e}$ a vector field on S, and $\beta : S \to]0, \frac{\pi}{2}[$ a function over S that verifies the following equation:*

$$\Delta(\beta) = -\tanh(\beta)(|\nabla\beta + 2(\cosh(\beta)^2 + \sinh(\beta)^2)\boldsymbol{e}|^2$$

then there exist one minimal immersion of S into $\mathbb{L}^3$ such that $\boldsymbol{e}$ is the characteristic vector fied, and β is the contact angle of this immersion.

2 The Contact Angle for minimal surface in the Lorentz Group $\mathbb{L}^3$

Consider in $\mathbb{C}^2$ the following objects:

- the Hermitian product: $(z,w) = z^1\bar{w}^1 - z^2\bar{w}^2$;
- the inner product: $\langle z,w\rangle = Re(z,w)$;
- the unit sphere: $L^3 = \{z \in \mathbb{C}^2 | (z,z) = -1\}$;
- the *Reeb* vector field in L^3, given by: $\xi(z) = iz$;
- the contact distribution in L^3, which is orthogonal to ξ:

$$\delta_z = \{v \in T_z L^3 | \langle \xi, v\rangle = 0\}.$$

Note that δ is invariant by the complex structure of $\mathbb{C}^2$.

Let now S be an immersed orientable surface in $\mathbb{L}^3$. Let (e_1, e_2) be a local frame of TS, where $e_1 \in TS \cap \Delta$. Then $\cos\beta = \langle \xi, e_2\rangle$.
Let e_1 be an unitary vector field in $TS \cap \Delta$, where Δ is the contact distribution. Thus follows that:

$$\begin{aligned} e_1 &= f_1 \\ e_2 &= \sinh(\beta)\, f_2 + \cosh(\beta)\, f_3 \\ e_3 &= \cosh(\beta)\, f_2 + \sinh(\beta)\, f_3 \end{aligned} \tag{2}$$

where β is the angle between f_3 and e_2, (e_1, e_2) are tangent to S and e_3 is normal to S

3 Equation for the Gaussian Curvature of a Minimal Surface in $\mathbb{L}^3$

In this section, we will give formulas for the Gaussian curvature of a minimal surface immersed in $\mathbb{L}^3$.
The reader can see [4], and [13] for further details.
Let $(\theta^1, \theta^2, \theta^3)$ be the coframe associated to (e_1, e_2, e_3).
We know that $\theta^3 = 0$ on S, then we obtain the following equation:

$$\cosh(\beta)\, w^3 = - \sinh(\beta) w^2 \tag{3}$$

we have also

$$\begin{aligned} w^2 &= \cosh\beta\theta^2 \\ w^3 &= -\sinh\beta\theta^2 \end{aligned}$$

It follows from (2) that:

$$\begin{aligned} d\theta^1 &+ \cosh(\beta) w_2^1 \wedge \theta^2 = 0 \\ d\theta^2 &+ \cosh(\beta)(w_1^2 - \sinh(\beta)\theta^2) \wedge \theta^1 = 0 \\ d\theta^3 &= d\beta \wedge \theta^2 + \sinh(\beta) w_2^1 \wedge \theta^1 + (1 + 3\cosh^2(\beta))\theta^1 \wedge \theta^2 \end{aligned}$$

Therefore the connection form of S is given by

$$\theta_1^2 \;=\; \cosh(\beta)(w_1^2 - \sinh(\beta)\theta^2) \tag{4}$$

Differentiating e_3 at the basis (e_1, e_2), we have fundamental second forms coeficients

$$De_3 \;=\; \theta_3^1 e_1 + \theta_3^2 e_2$$

where

$$\begin{aligned} \theta_3^1 &= \sinh(\beta) w_2^1 - \cosh^2(\beta)\theta^2 \\ \theta_3^2 &= d\beta + (\cosh^2(\beta) + \sinh^2(\beta))\theta^1 \end{aligned}$$

It follows from $d\theta^3 = 0$, that

$$w_2^1(e_2) \;=\; \frac{\beta_1}{\sinh\beta} + \frac{(-1 + 3\cosh^2\beta)}{\sinh\beta} \tag{5}$$

onde $d\beta(e_1) = \beta_1$.
The condition of minimality is equivalent to the following equation

$$\theta_1^3 \wedge \theta^2 - \theta_2^3 \wedge \theta^1 = 0$$

we have

$$w_2^1(e_1) \;= -\tfrac{\beta_2}{\sinh(\beta)} \tag{6}$$

where $d\beta(e_2) = \beta_2$.
It follows from (4), (5) and (6),

$$\begin{aligned} \theta_2^1 &= \coth(\beta)(\beta_2\theta^1 + (-\beta_1 + 2(1 - 2\cosh^2\beta))\theta^2) \\ \theta_3^1 &= -\beta_2\theta^1 + (\beta_1 + \cosh^2(\beta) + \sinh^2(\beta))\theta^2 \\ \theta_3^2 &= (\beta_1 + \cosh^2(\beta) + \sinh^2(\beta))\theta^1 + \beta_2\theta^2 \end{aligned}$$

Gauss equation is

$$d\theta_1^2 = \Omega_1^2 + \theta_2^3 \wedge \theta_1^3$$

which implies

$$d\theta_1^2 \;=\; 1 - |\nabla\beta|^2 - 2\beta_1(\cosh^2(\beta) + \sinh^2(\beta)) - (\cosh^2(\beta) + \sinh^2(\beta))^2 \tag{7}$$

where:

$$\Omega_1^2(e_2, e_1) = 1 \tag{8}$$

and therefore

$$K \;=\; 1 - |\nabla\beta + (\cosh^2(\beta) + \sinh^2(\beta))e_1|^2 \tag{9}$$

4 Main Result

4.1 Proof of the Theorem 1

When the contact angle β is constant, we have that: $K = 1 - (\cosh^2(\beta) + \sinh^2(\beta))^2$, we know that $(\cosh^2(\beta) + \sinh^2(\beta)) \geq 1$, and therefore $K \leq 0$.

4.2 Proof of the Theorem 2

Let S be an orientable surface in $\mathbb{L}^3$, and let $\mathbf{e}$ be an unit vector field on S. We choose an orthonormal positive basis (e_1, e_2) with $e_1 = e$, and let (θ^1, θ^2) be a coframe on S.
For each function $\beta : S \to]0, \frac{\pi}{2}[$ that satisfies the following Laplacian equation:

$$\Delta(\beta) \quad = \quad -\tanh(\beta)(|\nabla\beta + 2(\cosh(\beta)^2 + \sinh(\beta)^2)e_1|^2$$

We define the following fundamental second form:

$$\begin{aligned} \theta_1^3 &= (d\beta + \theta^1) \circ J \\ \theta_2^3 &= \quad - (d\beta + \theta^1) \end{aligned} \tag{10}$$

Now, the proof follows from Gauss-Codazzi equations. □.

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