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Characterization of Fuzzy Sub ℓ – Rings

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Abstract

The concept of level ℓ – subring of a fuzzy sub ℓ – ring is introduced. The relation between fuzzy sub ℓ – ring and ℓ – sub ring, characterization theorem for fuzzy sub ℓ – ring, the necessary and sufficient condition for the equality of two level ℓ – subrings in a family of level ℓ – subrings are obtained.

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1. Introduction

In 1965, Lofti. A. Zadeh [5] had initiated fuzzy set theory as a modification of the ordinary set theory. In 1982, Liu [3] developed the concept of fuzzy subrings as well as fuzzy ideals in rings. R. Natarajan and S. Mohanavalli [4] introduced the concept of fuzzy sub ℓ – ring of an ℓ – ring. In this paper, we made an attempt to characterize fuzzy sub ℓ – rings of an ℓ – ring through its level subsets.

2. Preliminaries

In this section we recall some basic definitions and results.

Definition: 2.1

A partial order set (L, ≤) is said to form a lattice if for every a, b ∈ L, Sup {a, b} and Inf {a, b} exist in L. In that case, we write Sup {a, b} = a ∨ b and Inf {a, b} = a ∧ b

A lattice L is called a distributive lattice if a∧(b ∨ c) = (a ∧ b)∨(a ∧ c), for all a, b, c ∈ L.

Definition: 2.2

A partial order ring is a ring R which is also a partial order set under a relation ≥, in which

(i) x ≥ y implies a + x ≥ a + y for all a ∈ R

(ii) x ≥ 0 and y ≥ 0 imply xy ≥ 0 in R.

Definition: 2.3

A lattice ordered ring or ℓ – ring R is a partial order ring in which any two elements x, y in R, have a least upper bound and a greatest lower bound.

Definition: 2.4

Let (R, +, ., ∨, ∧) be an ℓ – ring. A non – empty subset S of R is called an **ℓ – subring** of R if it satisfies the following conditions:

(i) x, y ∈ S ⇒ x ∨ y, x ∧ y ∈ S.

(ii) x, y ∈ S ⇒ x – y, xy ∈ S.

Definition: 2.5

Let X be a non – empty set. A mapping μ: X → [0, 1] is called a fuzzy subset of X.

Definition: 2.6

Let μ be any fuzzy subset of a set X and μ = (x_i, t_i) / i = 1 to n and t_i ∈ [0, 1]}. Then, {t_i / i =1 to n} is called the image set of μ and is denoted by Im μ.

Definition: 2.7

Let μ be any fuzzy subset of a set x and t ∈ Im μ. Then the set $\mu_t = \{x \in X / \mu(x) \geq t\}$ is called the level subset of μ.

3. Fuzzy Sub *ℓ* – Ring.

In this paper, R = (R, +, ., ∨, ∧) denotes the *ℓ* – ring with additive identity 0.

Definition: 3.1

A fuzzy subset μ of a lattice ordered ring (or *ℓ* – ring in short) R, is called a fuzzy sub *ℓ* – ring of R, if for all x, y ∈ R, the following conditions are satisfied:

(i) $\mu(x \vee y) \geq \min\{\mu(x), \mu(y)\}$

(ii) $\mu(x \wedge y) \geq \min\{\mu(x), \mu(y)\}$

(iii) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$

(iv) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$

Example: 3.2

Consider the fuzzy subset μ_1 of the *ℓ* – ring (Z, +, ., ∨, ∧).

$$\mu_1(x) = \begin{cases} .9 \; if \;\; x \in \langle 4 \rangle \\ .1 \quad Z \sim \langle 4 \rangle \end{cases}$$

Then μ_1 is a fuzzy sub *ℓ* – ring of Z.

4. Relation between fuzzy sub *ℓ* – ring and *ℓ* – subring.

This section investigates the relation between a non – empty subset of an *ℓ* – ring and the fuzzy sub *ℓ* – ring of the *ℓ* – ring, defined in terms of that subset. Then we show the converse relation is also true.

Theorem: 4.1

Let H be any non – empty subset of an *ℓ* – ring R, H ≠ R. Let μ be any fuzzy subset of R defined by, $\mu(x) = \begin{cases} s \; if \;\; x \in H \\ t \; if \;\; x \in R \sim H \end{cases}$, where s, t ∈[0, 1], s > t. Then μ is a fuzzy sub *ℓ* – ring of R if and only if H is an *ℓ* – subring of R.

Proof:

Let H be any non– empty subset of an *ℓ* – ring R, H ≠ R.

Let x, y $\in$ R be arbitrary.

Consider the function, $\mu(x) = \begin{cases} s \text{ if } x \in H \\ t \text{ if } x \in R \sim H \end{cases}$, where s, t $\in$ [0, 1], s > t.

Assume that μ is a fuzzy sub ℓ – ring of R.

Let a, b $\in$ H be arbitrary.

Then, μ(a) = s and μ(b) = s. And min {μ(a), μ(b)} = min{s, s} = s

Hence all the values of $\mu(a \vee b)$, $\mu(a \wedge b)$, μ(a – b) and μ(ab) are greater than or equal to s.

But μ has only two values s and t with s > t.

Therefore, all the values of $\mu(a \vee b)$, $\mu(a \wedge b)$, μ(a – b) and μ(ab) are equal to s.

This implies $a\vee b$, $a\wedge b$, a–b and ab $\in$ H.

This proves that H is an ℓ – sub ring of R.

Conversely, assume that H is any ℓ – sub ring of an ℓ – ring R, H $\neq$ R.

Let x, y $\in$ R be arbitrary.

Suppose x, y $\in$ H

Then, μ(x) = s, μ(y) = s and min {μ(x), μ(y)} = min {s, s} = s.

x, y $\in$ H, $x \vee y$, $x \wedge y$, x – y , xy $\in$ H, since H is an ℓ – sub ring of R.

$\Rightarrow$ $\mu(x\vee y)$ = s, $\mu(x\wedge y)$ = s, μ(x – y) = s, μ(xy) = s.

Then all the inequalities are satisfied in this case.

Suppose x, y $\in$ R ~ H

Then, μ(x) = t, μ(y) = t and min {μ(x), μ(y)} = min {t, t} = t.

Here $x\vee y$, $x\wedge y$, x – y, xy $\in$ R.

Then all the values of $\mu(x\vee y)$, $\mu(x\wedge y)$, μ(x – y) and μ(xy) are either t or s.

Then all the inequalities are satisfied in this case also.

Suppose x $\in$ H, y $\in$ R ~ H

μ(x) = s, μ(y) = t and min {μ(x), μ(y)} = min {s, t} = t.

Clearly $x\vee y$, $x\wedge y$, x – y, xy $\in$ R

Then all the values of $\mu(x\vee y)$, $\mu(x\wedge y)$, μ(x – y) and μ(xy) are either t or s.

Then all the inequalities are satisfied in this case also.

Thus μ is a fuzzy sub ℓ – ring of R.

Corollary: 4.2

If a non – empty subset H of an ℓ – ring R, is an ℓ – sub ring of R, then $¥_H$, the characteristic function of the subset H of R is a fuzzy sub ℓ – ring of R.

Proof: Take s =1, t = 0 in the above theorem.

5. Characterization of fuzzy sub ℓ – ring of an ℓ – ring

In this section, we characterize the fuzzy sub ℓ – ring of an ℓ – ring.

Theorem: 5.1

A fuzzy subset μ of an ℓ – ring R, is a fuzzy sub ℓ – ring of R if and only if, the level subsets μ_t, $t \in \text{Im}\ \mu$ are ℓ – subrings of R.

Proof:

Assume that μ is any fuzzy sub ℓ – ring of R; Let $t \in \text{Im}\ \mu$ be arbitrary.
Consider the level subset $\mu_t = \{x \in R / \mu(x) \geq t\}$.
Then, $0 \in \mu_t$, for all t.
That is $\mu_t \neq \phi$
Let $x, y \in \mu_t$.
Then, $\mu(x) \geq t$ and $\mu(y) \geq t$ and $\min\{\mu(x), \mu(y)\} \geq t$
$\Rightarrow \mu(x \vee y) \geq t$; $\mu(x \wedge y) \geq t$; $\mu(x - y) \geq t$ and $\mu(xy) \geq t$
$\Rightarrow x \vee y \in \mu_t$, $x \wedge y \in \mu_t$, $x - y \in \mu_t$, $xy \in \mu_t$.
Thus the level subsets μ_t, $t \in \text{Im}\ \mu$ are ℓ – sub rings of R.

Conversely, assume that the level subsets μ_t, $t \in \text{Im}\ \mu$ are ℓ – sub rings of R.
Let $x, y \in R$ be arbitrary.
Let $\min\{\mu(x), \mu(y)\} = r$
$\Rightarrow$ Either $\mu(x) = r$ and $\mu(y) \geq \mu(x) = r$ or $\mu(y) = r$ and $\mu(x) \geq \mu(y) = r$
$\Rightarrow \mu(x) \geq r$ and $\mu(y) \geq r$
$\Rightarrow x, y \in \mu_r$.
$\Rightarrow x \vee y, x \wedge y, x - y, xy \in \mu_r$, since μ_r is an ℓ – subring of R.
$\Rightarrow \mu(x \vee y) \geq r$; $\mu(x \wedge y) \geq r$; $\mu(x - y) \geq r$ and $\mu(xy) \geq r$
Thus μ is a fuzzy sub ℓ – ring of R.

Definition: 5.2

Let μ be any fuzzy sub ℓ – ring of R; $t \in [0, 1]$; and $t \leq \mu(0)$. Then the ℓ – subring μ_t of R is called a level ℓ – subring of μ.

Then $\mathbf{F}_\mu = \{\mu_t / t \in \text{Im}\ \mu\}$ is called as the family of level ℓ – sub rings of μ.

6. Equality of two level ℓ – sub rings in a family of level ℓ – sub rings

Theorem: 6.1

Two level ℓ – sub rings μ_s and μ_t of $\mathbf{F}_\mu$ (with $s < t$) of a fuzzy sub ℓ – ring μ of R are equal if and only if, there is no x in R such that $s \leq \mu(x) < t$.

Proof:

Let μ_s and μ_t be two level ℓ – subrings of a fuzzy sub ℓ – ring μ of R, with $s < t$
Assume that μ_s and μ_t are equal.

On the contrary, assume that $s \leq \mu(x) < t$ for some x in R.
$\Rightarrow x \in \mu_s$ and $x \notin \mu_t$
$\Rightarrow \mu_s \neq \mu_t$. and this is a contradiction to our assumption.

Conversely, assume that no x in R such that $s \leq \mu(x) < t$.
$\mu_s = \{x \in R \,/\, \mu(x) \geq s\}$ and $\mu_t = \{x \in R \,/\, \mu(x) \geq t\}$ and $s < t \Rightarrow \mu_t \subseteq \mu_s$.
Let $x \in \mu_s$.
Then $\mu(x) \geq s$.
Suppose $\mu(x) < t$. Then, $s \leq \mu(x) < t$. and this is a contradiction.
Hence $\mu(x) > t$. Then, $x \in \mu_t$.
$\Rightarrow \mu_s \subseteq \mu_t$.
Thus, $\mu_s = \mu_t$.
Hence two level ℓ – subrings are equal.

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