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Preorders and Various Operators

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Abstract

We study the relations among preorders, extensional systems, D-operators and J-operators. In particular, we investigated the functorial relations among them.

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1 Introduction and preliminaries

Rough set theory was introduced by Pawlak [4] to generalize the classical set theory. Rough approximations are defined by a partition of the universe which is corresponding to the equivalence relation about information. Järvinen et.al.[3] define rough approximations on preordered relations that are not necessarily equivalence relations. It is an important mathematical tool for data analysis and knowledge processing [1-6]. Yao [7,8] investigated the relation between the operators and rough approximations.

In this paper, we introduce the D-operator and J-operator. We investigated the relation between the operators and the general approximations. We study the relations among preorders, extensional systems, D-operators and J-operators. In particular, we investigated the functorial relations among them.

Let X be a set. A relation $e_X \subset X \times X$ is called a *preorder* if it is reflexive and transitive. If (X, e_X) is a preordered set and we define a relation $(x, y) \in e_X^{-1}$ iff $(y, x) \in e_X$, then (X, e_X^{-1}) is a preordered set.

2 Preorders and various operators

A function $D : P(X) \to P(X)$ is called a *D-operator* on X if it satisfies the following conditions:

(D1) $D(A) \subset A^c$,
(D2) If $A \subset B$, then $D(A) \supset D(B)$
(D3) $D(A)^c \subset D(D(A))$.

The pair (X, D) is called a *D-space*. Let (X, D_X) and (Y, D_Y) be D-spaces. A map $f : (X, D_X) \to (Y, D_Y)$ is called a *D-map* if $f(D_X(A)) \subset D_Y(f(A))$ for each $A \in P(X)$. Let D_1 and D_2 be D-operators on X. D_2 is coarser than D_1 if $D_1 \subset D_2$.

A function $J : P(X) \to P(X)$ is called a *J-operator* on X if it satisfies the following conditions:
(J1) $A^c \subset J(A)$,
(J2) If $A \subset B$, then $J(A) \supset J(B)$
(J3) $J(J(A)) \subset J(A)^c$.

The pair (X, J) is called a *J-space*. Let (X, J_X) and (Y, J_Y) be J-spaces. A map $f : (X, J_X) \to (Y, J_Y)$ is called a *J-map* if $f^{-1}(J_Y(B)) \supset J_X(f^{-1}(B))$ for each $B \in P(Y)$. Let J_1 and J_2 be J-operators on X. J_2 is coarser than J_1 if $J_1 \subset J_2$.

A family $\mathcal{F} \subset P(X)$ is called an *extensional system* on X if $A_i^c, \bigcap_{i\in\Gamma} A_i \in \mathcal{F}$ for each $A_i \in \mathcal{F}$.

Theorem 2.1 *Let R be a reflexive relation on X such that $(x, y) \in R$ and $(x, z) \in R$ implies $(y, z) \in R$. We define*

$$[[R^c]](A) = \{x \in X \mid (\forall y \in Y)(y \in A \to (x, y) \in R^c)\}.$$

Then $[[R^c]]$ is a D-operator with $[[R^c]](\bigcup_{i\in\Gamma} A_i) = \bigcap_{i\in\Gamma}[[R^c]](A_i)$ for $A_i \subset X$.

Proof. (D1) If $x \in A$, then $x \in A\ \&(x, x) \notin R^c$ iff $x \in [[R^c]](A)^c$. Hence $[[R^c]](A) \subset A^c$.

(D2) It follows by the definition of $[[R^c]]$.

(D3) Let $x \notin [[R^c]]([[R^c]](A))$ iff $\vdash (\exists y)(y \in [[R^c]](A)\ \&(x, y) \in R)$ iff $\vdash (\exists y)((\forall z \in X)(z \in A \to (y, z) \in R^c)\ \&\ (x, y) \in R)$ iff $\vdash (\exists y)((\forall z \in X)((y, z) \in R \to z \notin A)\ \&\ (x, y) \in R$. Since $((y, z) \in R \to z \notin A)\ \&\ (x, y) \in R\&\ (x, z) \in R$ implies $((y, z) \in R \to z \notin A)\ \&\ (y, z) \in R$ implies $z \notin A$, we have

$$((y, z) \in R \to z \notin A)\ \&\ (x, y) \in R\&\ (x, z) \in R \to z \notin A$$

$$((y, z) \in R \to z \notin A)\ \&\ (x, y) \in R \to (x, z) \in R \to z \notin A$$

By M.P., $\vdash (\forall z \in X)((x, z) \in R \to z \notin A)$. Thus, $x \in [[R^c]](A)$.

$$\begin{aligned} x \notin [[R^c]](\textstyle\bigcup_{i\in\Gamma} A_i) \quad & \text{iff } (\exists y \in Y)(y \in \textstyle\bigcup_{i\in\Gamma} A_i\ \&\ (x, y) \in R) \\ & \text{iff } (\exists y \in Y)(\exists i \in \Gamma)(y \in A_i\ \&\ (x, y) \in R) \\ & \text{iff } (\exists i \in \Gamma)(\exists y \in Y)(y \in A_i\ \&\ (x, y) \in R) \\ & \text{iff } x \notin \textstyle\bigcap_{i\in\Gamma}[[R^c]](A_i). \end{aligned}$$

Theorem 2.2 *Let (X, D) be a D-space. Define $(x, y) \in e_D$ iff $x \in D(\{y\})^c$. Then:* (1) *$D(D(A)) = D(A)^c$ and $D(D(A)^c) = D(A)$.*
(2) *$\mathcal{D} = \{A \in L^X \mid D(A) = A^c\}$ is an extensional system.*
(3) *e_D is a preorder on X.*
(4) *If $D(\cup_{i\in\Gamma} A_i) = \cap_{i\in\Gamma} D(A_i)$ for each family $\{A_i \mid i \in \Gamma\}$, then $D = [[e^c_D]]$.*

Proof. (1) By (D1), $D(D(A)) \subset D(A)^c$. By (D3), $D(D(A)) = D(A)^c$.

Since $D(A)^c \subset D(D(A))$, $D(D(A)^c) \supset D(D(D(A))) \supset (D(D(A)))^c = D(A)$ and $D(D(A)^c) \subset D(A)^c$.

(2) Let $A_i \in \mathcal{D}$ for all $i \in \Gamma$. Then $D(\bigcap_{i\in\Gamma} A_i) \subset \bigcup_{i\in\Gamma} A^c_i$. Since $A_i \supset \bigcap_{i\in\Gamma} A_i$, then $A^c_i = D(A_i) \subset D(\bigcap_{i\in\Gamma} A_i)$. So, $\bigcup_{i\in\Gamma} A^c_i \subset D(\bigcap_{i\in\Gamma} A_i)$. Hence $\bigcap_{i\in\Gamma} A_i \in \mathcal{D}$.

Let $A \in \mathcal{D}$. Then $D(A) = A^c$. $D(A^c) = D(D(A)) = D(A)^c = A$. Thus $A^c \in \mathcal{D}$.

(3) Since $x \in \{x\} \subset D(\{x\})^c$ from (D1), $(x, x) \in e_D$. Let $(x, y) \in e_D$ and $(y, z) \in e_D$. Since $x \in D(\{y\})^c$ and $y \in D(\{z\})^c$ iff $D(\{y\}) \subset \{x\}^c$ and $D(\{z\}) \subset \{y\}^c$, we have $D(\{z\}) = D(D(\{z\})^c) \subset D(\{y\}) \subset \{x\}^c$. Thus, $(x, z) \in e_D$.

(4)

$$\begin{aligned} y \in D(A) = D(\textstyle\bigcup_{x\in A}\{x\}) \quad &\text{iff } y \in \textstyle\bigcap_{x\in A} D(\{x\}) \\ &\text{iff } (\forall x \in X)(x \in A \to y \in D(\{x\})) \\ &\text{iff } (\forall x \in X)(x \in A \to (y, x) \in e^c_D) \\ &\text{iff } y \in [[e^c_D]](A). \end{aligned}$$

Hence $D(A) = [[e^c_D]](A)$ for each $A \in P(X)$.

Theorem 2.3 *Let $f : X \to Y$ be a function and (Y, D_Y) a D-space. Then*
(1) *$f^{\Leftarrow}(D_Y)$ is the coarsest D-operator on X which f is a D-map where $f^{\Leftarrow}(D_Y)(A) = f^{-1}(D_Y(f(A)))$ for each $A \subset X$.*
(2) *$e_{f^{\Leftarrow}(D_Y)} = (f \times f)^{-1}(e_{D_Y})$.*
(3) *$\mathcal{D}_{f^{\Leftarrow}(D_Y)} \subset f^{-1}(\mathcal{D}_{D_Y}) = \{f^{-1}(B) \mid D_Y(B) = B^c\}$.*
(4) *If f is onto, then $\mathcal{D}_{f^{\Leftarrow}(D_Y)} = f^{-1}(\mathcal{D}_{D_Y})$.*
(5) *If $D_Y(\bigcup_{i\in\Gamma} B_i) = \bigcap_{i\in\Gamma} D_Y(B_i)$, then $f^{\Leftarrow}(D_Y)(\bigcup_{i\in\Gamma} B_i) = \bigcap_{i\in\Gamma} f^{\Leftarrow}(D_Y)(B_i)$ and*

$$[[(e_{f^{\Leftarrow}(D_Y)})^c]] = [[((f \times f)^{-1}(e_{D_Y}))^c]] = f^{\Leftarrow}(D_Y).$$

Proof. (1) (D1) $f^{\Leftarrow}(D_Y)(A) = f^{-1}(D_Y(f(A))) \subset f^{-1}(f(A)^c) \subset A^c$.

(D2)

$$\begin{aligned} f^{\Leftarrow}(D_Y)\Big((f^{\Leftarrow}(D_Y)(A))\Big) \quad &= f^{\Leftarrow}(D_Y)\Big((f^{-1}(D_Y(f(A))))\Big) \\ &= f^{-1}(D_Y(f(f^{-1}(D_Y(f((A))))))) \\ &\supset f^{-1}(D_Y(D_Y(f(A)))) \\ &\supset f^{-1}(D_Y(f(A))^c) = (f^{-1}(D_Y(f(A))))^c \\ &\supset (f^{\Leftarrow}(D_Y)(A))^c. \end{aligned}$$

Since $f(f^{\Leftarrow}(D_Y)(A)) = f(f^{-1}(D_Y(f(A)))) \subset D_Y(f(A))$, then $f : (X, f^{\Leftarrow}(D_Y)) \to (Y, D_Y)$ is a D-map. Finally, if $f : (X, D_1) \to (Y, D_Y)$ is a D-map, then $f(D_1(A)) \subset D_Y(f(A))$. It implies

$$D_1(A) \subset f^{-1}(f(D_1(A))) \subset f^{-1}(D_Y(f(A))) = f^{\Leftarrow}(D_Y)(A)$$

Hence $D_1 \subset f^{\Leftarrow}(D_Y)$.

(2) We have $e_{f^{\Leftarrow}(D_Y)} = (f \times f)^{-1}(e_{D_Y})$ from:

$$\begin{array}{ll} (x,y) \in e_{f^{\Leftarrow}(D_Y)} & \text{iff } x \in f^{\Leftarrow}(D_Y)(\{y\})^c \\ \text{iff } x \in f^{\Leftarrow}(D_Y)(\{y\})^c & \text{iff } f(x) \in D_Y(f(\{x\}))^c \\ \text{iff } f(x) \in D_Y(\{f(y)\})^c & \text{iff } (f(x), f(y)) \in e_{D_Y} \\ \text{iff } (x,y) \in (f \times f)^{-1}(e_{D_Y}). & \end{array}$$

(3) Let $A \in \mathcal{D}_{f^{\Leftarrow}(D_Y)}$. Then $A^c = f^{\Leftarrow}(D_Y)(A) = f^{-1}(D_Y(f(A)))$ implies $A = f^{-1}((D_Y(f(A)))^c)$. Since $D_Y((D_Y(f(A)))^c) = D_Y(f(A))$, $A \in f^{-1}(\mathcal{D}_{D_Y})$.

(4) Let $A \in f^{-1}(\mathcal{D}_{D_Y})$. Then there exists $B \in \mathcal{D}_{D_Y}$ such that $A = f^{-1}(B)$ with $B^c = D_Y(B)$. Since f is onto, $f(A) = f(f^{-1}(B)) = B$. So,

$$A^c = f^{-1}(B^c) = f^{-1}(D_Y(B)) = f^{-1}(D_Y(f(A))) = f^{\Leftarrow}(D_Y)(A)$$

Thus, $A \in \mathcal{D}_{f^{\Leftarrow}(D_Y)}$.

(5) $f^{\Leftarrow}(D_Y)(\bigcup_{i\in\Gamma} B_i) = f^{-1}(D_Y(f(\bigcup_{i\in\Gamma} B_i))) = \bigcap_{i\in\Gamma} f^{\Leftarrow}(D_Y)(B_i)$. By Theorem 2.2(4), the results hold.

From Theorems 2.1 and 2.3, we can obtain the following corollary.

Corollary 2.4 *Let $f : X \to Y$ be a function and R_Y a reflexive relation on Y such that $(x, y) \in R$ and $(x, z) \in R$ implies $(y, z) \in R$. Then*

(1) *$f^{\Leftarrow}([[R_Y^c]])$ is the coarsest D-operator on X which f is a D-map.*

(2) *$e_{f^{\Leftarrow}([[R_Y^c]])} = (f \times f)^{-1}([[R_Y^c]])$.*

(3) *$\mathcal{D}_{f^{\Leftarrow}([[R_Y^c]])} \subset f^{-1}(\mathcal{D}_{[[R_Y^c]]})$.*

(4) *If f is onto, then $\mathcal{D}_{f^{\Leftarrow}([[R_Y^c]])} = f^{-1}(\mathcal{D}_{[[R_Y^c]]})$.*

(5) *$[[e^c_{f^{\Leftarrow}([[R_Y^c]])}]] = [[(f \times f)^{-1}([[R_Y^c]])^c]] = f^{\Leftarrow}([[R_Y^c]])$.*

Example 2.5 Let $X = \{a, b, c, d\}$ and $Y = \{x, y, z\}$ be sets and $f(a) = f(b) = x, f(c) = y, f(d) = z$. Define $D : P(X) \to P(Y)$ as follows:

$$D(\emptyset) = Y,\ D(\{x, y\}) = \{z\},\ D(\{y, z\}) = D(\{x, z\}) = \emptyset,$$

$$D(\{y\}) = D(\{x\}) = \{z\}, D(\{z\}) = \{x, y\}, D(Y) = \emptyset.$$

We obtain:

$$e_D = \{(x, x), (x, y), (y, x), (y, y), (z, z)\}.$$

Since $D(\cup_i B_i) = \cap_i D(B_i)$, $D = [[e^c_D]]$. We obtain:

$$f^{\Leftarrow}(D)(\{a\}) = f^{-1}(D(f(\{a\})) = \{d\} = f^{\Leftarrow}(D)(\{b\}) = f^{\Leftarrow}(D)(\{c\}), f^{\Leftarrow}(D)(\emptyset) = Y,$$

$$f^{\Leftarrow}(D)(\{d\}) = \{a,b,c\}, f^{\Leftarrow}(D)(\{a,b\}) = \{d\} = f^{\Leftarrow}(D)(\{a,c\}) = f^{\Leftarrow}(D)(\{b,c\}),$$

$$f^{\Leftarrow}(D)(\{a,d\}) = Y = f^{\Leftarrow}(D)(\{b,d\}) = f^{\Leftarrow}(D)(\{c,d\}), f^{\Leftarrow}(D)(\{a,b,c\}) = \{d\},$$

$$f^{\Leftarrow}(D)(\{a,b,d\}) = \emptyset = f^{\Leftarrow}(D)(\{a,c,d\}) = f^{\Leftarrow}(D)(\{b,c,d\}) = f^{\Leftarrow}(D)(X),$$

$$\begin{aligned} e_{f^{\Leftarrow}(D)} &= \{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c),(d,d)\} \\ &= (f\times f)^{-1}(e_D). \end{aligned}$$

Theorem 2.6 *Let R be a reflexive relation on X such that $(x,y) \in R$ and $(x,z) \in R$ implies $(y,z) \in R$. We define*

$$\langle R\rangle^c(A) = \{x \in X \mid (\exists y \in X)(y \in A^c \,\&\, (x,y) \in R)\}.$$

Then $\langle R\rangle^c$ is a J-operator with $\langle R\rangle^c(\bigcap_{i\in\Gamma} A_i) = \bigcup_{i\in\Gamma}\langle R\rangle^c(A_i)$ for $A_i \subset X$.

Proof. (J1) If $x \in A^c$, then $x \in A^c$ $(x,x) \in R$. Hence $x \in \langle R\rangle^c(A)$.

(J3) Let $x \in \langle R\rangle^c(\langle R\rangle^c(A))$ iff $\vdash (\exists y \in X)(y \in (\langle R\rangle^c(A))^c \,\&\, (x,y) \in R)$ iff $\vdash (\exists y \in X)((x,y) \in R \,\&\, (\forall z \in X)((y,z) \in R \to z \in A))$ iff $\vdash (\exists y \in X)(\forall z \in X)((x,y) \in R \,\&\, ((y,z) \in R \to z \in A))$. Since $\vdash ((x,y) \in R \,\&\, (x,z) \in R \& \, ((y,z) \in R \to z \in A) \to (y,z) \in R \,\&\, ((y,z) \in R \to z \in A))$ and $\vdash ((y,z) \in R \,\&\, ((y,z) \in R \to z \in A) \to z \in A)$, by M.P, then $\vdash ((x,y) \in R \,\&\, (x,z) \in R \& \, ((y,z) \in R \to z \in A) \to z \in A))$. Hence $\vdash ((x,y) \in R \,\&\, ((y,z) \in R \to z \in A) \to ((x,z) \in R \to z \in A))$. So, $\vdash (\forall z \in)((x,z) \in R \to z \in A)$ iff $x \in (\langle R\rangle^c(A))^c$.

$$\begin{aligned} x \in \langle R\rangle^c(\textstyle\bigcap_{i\in\Gamma} A_i) \quad & \text{iff } (\exists y \in X)(y \in (\textstyle\bigcap_{i\in\Gamma} A_i)^c \,\&\, (x,y) \in R) \\ & \text{iff } (\exists x \in X)(\exists i \in \Gamma)(x \in A_i^c \,\&\, (x,y) \in R) \\ & \text{iff } (\exists i \in \Gamma)(x \in \langle R\rangle^c(A_i) \\ & \text{iff } x \in \textstyle\bigcup_{i\in\Gamma}\langle R\rangle^c(\bigcap_{i\in\Gamma} A_i). \end{aligned}$$

Theorem 2.7 *Let (X,J) be a J-space. Define $(x,y) \in e_J$ iff $x \in J(\{y\}^c)$.*
Then (1) *$J(J(A)) = J(A)^c$ and $J(J(A)^c) = J(A)$.*
(2) *$\mathcal{J} = \{A \in P(X) \mid J(A) = A^c\}$ is an external system.*
(3) *e_J is a preorder on X.*
(4) *If $J(\cap_{i\in\Gamma} A_i) = \cup_{i\in\Gamma} J(A_i)$ for each family $\{A_i \mid i \in \Gamma\}$, then $J = \langle e_J\rangle^c$.*
(5) *Define $D(A) = J(A^c)^c$ for all $A \subset X$. Then D is a D-operator.*

Proof. (1) By (J1), $J(J(A)) \supset J(A)^c$. By (J3), $J(J(A)) = J(A)^c$.

Since $J(A)^c \supset J(J(A))$, $J(J(A)^c) \subset J(J(J(A))) \subset (J(J(A)))^c = J(A)$ and $J(J(A)) \supset J(A)^c$.

(2) Let $A_i \in \mathcal{J}$ for all $i \in \Gamma$. Then $J(\bigcup_{i\in\Gamma} A_i) \supset \bigcap_{i\in\Gamma} A_i^c$. Since $A_i \subset \bigcup_{i\in\Gamma} A_i$, then $A_i^c = J(A_i) \supset J(\bigcup_{i\in\Gamma} A_i)$. So, $\bigcap_{i\in\Gamma} A_i^c \supset J(\bigcup_{i\in\Gamma} A_i)$. Hence $\bigcup_{i\in\Gamma} A_i \in \mathcal{J}$.

Let $A \in \mathcal{J}$. Then $J(A) = A^c$. $J(A^c) = J(J(A)) = J(A)^c = A$. Thus $A^c \in \mathcal{J}$.

(3) Since $x \in \{x\} \subset J(\{x\}^c)$ from (J1), $(x,x) \in e_J$. Let $(x,y) \in e_J$ and $(y,z) \in e_J$. Since $x \in J(\{y\}^c)$ and $y \in J(\{z\}^c)$ iff $x \in J(\{y\}^c)$ and $\{y\} \subset J(\{z\}^c)$, we have

$$x \in J(\{y\}^c) \subset J(J(\{z\}^c)^c) = J(\{z\}^c)$$

Thus $(x,z) \in e_J$.

(4) Since $A = \cap_{x\in A^c}\{x\}^c$, $J(A) = J(\cap_{x\in A^c}\{x\}^c) = \cup_{x\in A^c} J(\{x\}^c)$. Thus

$$\begin{aligned} y \in J(A) \quad & \text{iff } y \in \cup_{x\in A^c} J(\{x\}^c) \\ & \text{iff } (\exists x \in X)(x \in A^c \ \&\ \ y \in J(\{x\}^c)) \\ & \text{iff } (\exists x \in X)(x \in A^c \ \&\ \ (y,x) \in e_J) \\ & \text{iff } y \in \langle e_J \rangle^c(A). \end{aligned}$$

(5) (D1) $D(A) = J(A^c)^c \subset A^c$. (D2) If $A \subset B$, then $J(A^c) \subset J(B^c)$. Hence $D(A) \supset D(B)$. (D3) $D(D(A)) = J(J(A^c))^c \supset J(A^c) = D(A)^c$.

Theorem 2.8 *Let $\mathcal{F}$ be an extensional system on X. Then*

(1) *Define $D(A) = \cup\{F \in \mathcal{F} \mid F \subset A^c\}$ for $A \subset X$. Then D is a D-operator.*

(2) *Define $J(A) = \cup\{F \in \mathcal{F} \mid A^c \subset F\}$ for $A \subset X$. Then J is a J-operator.*

Proof. (1) (D1) and (D2) are easily proved.

(D3) Since $D(D(A)) = \cup\{F \in \mathcal{F} \mid F \subset D(A)^c\}$ and $D(A)^c \in \mathcal{F}$, $D(D(A)) \supset D(A)^c$.

(2) is similarly proved as in (1).

Example 2.9 Let $X = \{x,y,z\}$ be a set.

(1) Let $R = \{(x,x),(x,y),(x,z),(y,y),(y,z),(z,z)\}$ be a relation. Since $(x,z) \in R$ and $(x,y) \in R$, but $(z,y) \notin R$, it does not satisfy the condition of Theorems 2.1 and 2.6. We obtain $[[R^c]], \langle R\rangle^c : P(X) \to P(X)$ as follows:

$$[[R^c]](\emptyset) = X,\ [[R^c]](\{x\}) = \{y,z\},\ [[R^c]](\{y\}) = \{z\}, [[R^c]](\{x,y\}) = \{z\},$$

$$[[R^c]](\{z\}) = [[R^c]](\{y,z\}) = \emptyset = [[R^c]](\{x,z\}) = [[R^c]](X),$$

$$\langle R\rangle^c(\emptyset) = X = \langle R\rangle^c(\{x\}) = \langle R\rangle^c(\{y\}) = \langle R\rangle^c(\{x,y\}).$$

$$\langle R\rangle^c(\{y,z\}) = \{x\}, \langle R\rangle^c(\{z\}) = \langle R\rangle^c(\{x,z\}) = \{x,y\}, \langle R\rangle^c(X) = \emptyset.$$

$$\{x\} = ([[R^c]](\{x\}))^c \not\subset [[R^c]]([[R^c]](\{x\})) = \emptyset.$$

$$X = \langle R\rangle^c(\langle R\rangle^c(\{y,z\})) \not\subset (\langle R\rangle^c(\{y,z\}))^c = \{y,z\}.$$

(2) Let $R = \{(x,x),(y,y),(y,z),(z,y),(z,z)\}$ be a relation. It satisfies the condition of Theorems 2.1 and 2.6. We obtain D-operator and J-operator $[[R^c]], \langle R\rangle^c : P(X) \to P(X)$ as follows:

$$[[R^c]](\emptyset) = X,\ [[R^c]](\{x\}) = \{y,z\},\ [[R^c]](\{y\}) = [[R^c]](\{z\}) = \{x\}, [[R^c]](\{x,y\}) = \emptyset,$$

$$[[R^c]](\{y,z\}) = \{x\}, \emptyset = [[R^c]](\{x,z\}) = [[R^c]](X),$$

$$\langle R\rangle^c(\emptyset) = \langle R\rangle^c(\{y\}) = \langle R\rangle^c(\{z\}) = X, \langle R\rangle^c(\{x\}) = \langle R\rangle^c(\{x,y\}) = \{y,z\},$$

$$\langle R\rangle^c(\{y,z\}) = \{x\}, \langle R\rangle^c(\{x,z\}) = \{y,z\}, \langle R\rangle^c(X) = \emptyset.$$

By Theorems 2.2 and 2.7, we can obtain a preorder $e_{[[R^c]]} = R = e_{\langle R\rangle^c}$ and extensional system $\mathcal{D} = \mathcal{J} = \{\emptyset, X, \{x\}, \{y,z\}\}$. Furthermore, $[[R^c]](\bigcup_{i\in\Gamma} A_i) = \bigcap_{i\in\Gamma}[[R^c]](A_i)$ and $\langle R\rangle^c(\bigcap_{i\in\Gamma} A_i) = \bigcup_{i\in\Gamma}\langle R\rangle^c(A_i)$ for $A_i \subset X$.

(3) Since $\mathcal{D} = \mathcal{J} = \{\emptyset, X, \{x\}, \{y,z\}\}$ in (2), by Theorem 2.8, we obtain D-operator and J-operator $D_{\mathcal{D}} = [[R^c]], J_{\mathcal{J}}\langle R\rangle^c$.

Theorem 2.10 *Let $f : X \to Y$ be a function and (Y, J_Y) a J-space. Then*

(1) $f^{\triangleleft}(J_Y)$ is the coarsest J-operator on X which f is a J-map where $f^{\triangleleft}(J_Y)(A) = f^{-1}(J_Y(f(A^c)^c))$ for $A \subset X$.

(2) $e_{f^{\triangleleft}(J_Y)} = (f\times f)^{-1}(e_{J_Y})$.

(3) $\mathcal{J}_{f^{\triangleleft}(J_Y)} \subset f^{-1}(\mathcal{J}_{J_Y}) = \{f^{-1}(B) \mid B^c = J_Y(B)\}$.

(4) If f is onto, then $\mathcal{J}_{f^{\triangleleft}(J_Y)} = f^{-1}(\mathcal{J}_{J_Y})$.

(5) If $J_Y(\cap_{i\in\Gamma} B_i) = \cup_{i\in\Gamma} J_Y(B_i)$, then $f^{\triangleleft}(J_Y)(\cap_{i\in\Gamma} A_i) = \cup_{i\in\Gamma} f^{\triangleleft}(J_Y)(A_i)$ and

$$\langle e_{f^{\triangleleft}(J_Y)}\rangle^c = \langle (f\times f)^{-1}(e_{J_Y})\rangle^c = f^{\triangleleft}(J_Y).$$

Proof. (1) (J1) $f^{\triangleleft}(J_Y)(A) = f^{-1}(J_Y(f(A^c)^c)) \supset f^{-1}(f(A^c)) \supset A^c$.

(J2)

$$\begin{aligned}
f^{\triangleleft}(J_Y)\Big((f^{\triangleleft}(J_Y)(A))\Big) &= f^{\triangleleft}(J_Y)\Big((f^{-1}(J_Y(f(A^c)^c)))\Big) \\
&= f^{-1}(J_Y(f\Big((f^{-1}(J_Y f(A^c)^c)^c\Big)^c)) \\
&\subset f^{-1}(J_Y(J_Y(f(A^c)^c))) \\
&\subset f^{-1}((J_Y(f(A^c)^c))^c \\
&= (f^{\triangleleft}(J_Y)(A))^c.
\end{aligned}$$

Since $f^{\triangleleft}(J_Y)(f^{-1}(B)) = f^{-1}(J_Y(f(f^{-1}(B^c))^c)) \subset f^{-1}(J_Y(B))$, then $f : (X, f^{\triangleleft}(J_Y)) \to (Y, J_Y)$ is a J-map. Finally, if $f : (X, J_1) \to (Y, J_Y)$ is a J-map, then $J_1(f^{-1}(B)) \subset f^{-1}(J_Y(B))$. Put $B = f(A^c)^c$. Then

$$J_1(A) \subset J_1(f^{-1}(f(A^c)^c)) \subset f^{-1}(J_Y(f(A^c)^c)) = f^{\triangleleft}(J_Y)(A)$$

Hence $J_1 \subset f^{\triangleleft}(J_Y)$.

(2) We have $e_{f^{\triangleleft}(J_Y)} = (f \times f)^{-1}(e_{J_Y})$ from:

$$\begin{aligned}(x,y) \in e_{f^{\triangleleft}(J_Y)} \quad & \text{iff } x \in f^{\triangleleft}(J_Y)(\{y\}^c) \\ & \text{iff } x \in f^{\triangleleft}(J_Y)(\{y\}^c) \\ & \text{iff } f(x) \in J_Y(f(\{x\})^c) \\ & \text{iff } f(x) \in J_Y(\{f(y)\}^c) \\ & \text{iff } (f(x), f(y)) \in e_{J_Y} \\ & \text{iff } (x,y) \in (f \times f)^{-1}(e_{J_Y}).\end{aligned}$$

(3) Let $A \in \mathcal{J}_{f^{\triangleleft}(J_Y)}$. Then $A^c = f^{\triangleleft}(J_Y)(A) = f^{-1}(J_Y(f(A^c)^c))$ implies $A = f^{-1}((J_Y(f(A^c)^c))^c)$. Since $J_Y((J_Y(f(A^c)^c))^c) = J_Y(f(A^c)^c)$, $A \in f^{\triangleleft}(\mathcal{J}_{J_Y})$.

(4) Let $A \in f^{\triangleleft}(\mathcal{J}_{J_Y})$. Then there exists $B \in P(Y)$ such that $A = f^{-1}(B)$ with $B^c = J_Y(B)$. Since f is onto, $f(A^c)^c = f(f^{-1}(B^c))^c = B$. So,

$$A^c = f^{-1}(B^c) = f^{-1}(J_Y(B)) = f^{-1}(J_Y(f(A^c)^c)) = f^{\triangleleft}(J_Y)(A)$$

Thus, $A \in \mathcal{J}_{f^{\triangleleft}(J_Y)}$.

(5) Since $f^{\triangleleft}(J_Y)(\cap_{i\in\Gamma} A_i) = f^{-1}(J_Y(f(f^{-1}(\cup_{i\in\Gamma} A_i^c))^c)) = \cup_{i\in\Gamma} f^{\triangleleft}(J_Y)(A_i)$, by Theorem 2.7 (4), the results hold.

Example 2.11 Let $X = \{a,b,c,d\}$ and $Y = \{x,y,z\}$ be sets and $f(a) = f(b) = x, f(c) = y, f(d) = z$. Define $J : P(Y) \to P(Y)$ as follows:

$$J(\emptyset) = Y,\ J(\{x,y\}) = \{y,z\},\ J(\{y,z\}) = \{x\}, J(\{x,z\}) = \{y,z\}.$$

$$J(\{y\}) = Y,\ J(\{x\}) = \{y,z\}, J(\{z\}) = Y, J(Y) = \emptyset.$$

We obtain:

$$e_J = \{(x,x),(y,y),(y,z),(z,y),(z,z)\}.$$

Since $J(\bigcap A_i) = \bigcup J(A_i)$, $J = \langle e_J \rangle^c$. We obtain:

$$f^{\triangleleft}(J)(\{a\}) = f^{-1}(J(f(\{a\}^c)^c) = Y = f^{\triangleleft}(J)(\{b\}) = f^{\triangleleft}(J)(\{c\}) = f^{\triangleleft}(J)(\emptyset),$$

$$f^{\triangleleft}(J)(\{d\}) = Y, f^{\triangleleft}(J)(\{a,d\}) = Y, f^{\triangleleft}(J)(\{a,b\}) = \{c,d\}, f^{\triangleleft}(J)(\{a,c\}) = \emptyset,$$

$$f^{\triangleleft}(J)(\{b,c\}) = \emptyset = f^{\triangleleft}(J)(\{b,d\}), f^{\triangleleft}(J)(\{c,d\}) = \{a,b\},$$

$$f^{\triangleleft}(J)(\{a,b,c\}) = \{c,d\}, f^{\triangleleft}(J)(\{a,b,d\}) = \{c,d\}, f^{\triangleleft}(J)(\{a,c,d\}) = \{a,b\},$$

$$f^{\triangleleft}(J)(\{b,c,d\}) = \{a,b\}, f^{\triangleleft}(J)(X) = X.$$

$$\begin{aligned} e_{f^{\triangleleft}(J)} &= \{(a,a),(a,b),(b,a),(b,b),(c,c),(c,d),(d,c),(d,d)\} \\ &= (f \times f)^{-1}(e_J).\end{aligned}$$

From Theorems 2.6 and 2.9, we can obtain the following corollary.

Corollary 2.12 *Let $f : X \to Y$ be a function and R_Y a reflexive relation on X such that $(x, y) \in R$ and $(x, z) \in R$ implies $(y, z) \in R$. Then*
(1) $f^{\triangleleft}(\langle R_Y\rangle^c)$ *is the coarsest J-operator on X which f is a J-map.*
(2) $e_{f^{\triangleleft}(\langle R_Y\rangle^c)} = (f \times f)^{-1}(\langle R_Y\rangle^c)$.
(3) $\mathcal{J}_{f^{\triangleleft}(\langle R_Y\rangle^c)} \subset f^{-1}(\mathcal{J}_{\langle R_Y\rangle^c}) = \{f^{-1}(B) \mid B^c = \langle R_Y\rangle^c(B)\}$.
(4) *If f is onto, then* $\mathcal{J}_{f^{\triangleleft}(\langle R_Y\rangle^c)} = f^{-1}(\mathcal{J}_{\langle R_Y\rangle^c})$.
(5) $\langle e_{f^{\triangleleft}(\langle R_Y\rangle^c)}\rangle^c = \langle (f \times f)^{-1}(\langle R_Y\rangle^c)\rangle^c = f^{\triangleleft}(\langle R_Y\rangle^c)$.

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