On Some Fuzzy Convex Invariants in a Fuzzy Convex Product Space

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Abstract

J. Eckhoff introduced the concept of convex product space in 1968. The classical convex invariants namely Helly number, Caratheodory number, Radon number and exchange number for the convex product space are determined and studied by G. Sierksma [5] in 1976. In [4], the authors studied some of these classical convex invariants in the fuzzy context and arrived at certain results. In this paper, the exchange number of a fuzzy convex product space with two factors is discussed in various situations.

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1 Introduction

This paper is a continuation of the study of fuzzy convex invariants in a fuzzy convex product space with two factors [4]. In [4], the Caratheodory number of a fuzzy convex product space with two factors is derived in connection with the exchange number of the factor spaces. In this paper, the exchange number of a convex product space with two factors is studied in the fuzzy context and some relations are established.
2 Basic definitions and properties

Definition 2.1 [1] A fuzzy subset $A$ on a nonempty set $X$ is a function from $X$ to $I = [0, 1]$. The set of all fuzzy subsets of $X$ is denoted by $I^X$.

A fuzzy point $a_{\alpha}$ is a fuzzy subset defined as

$$a_{\alpha}(y) = \begin{cases} \alpha > 0, & \text{if } y = a \\ 0, & \text{otherwise} \end{cases}$$

Definition 2.2 [1] Support of a fuzzy subset $A$ is $\text{Supp}(A) = \{x \in X | A(x) > 0\}$.

A fuzzy subset is said to be finite if its support is finite.

Definition 2.3 [1] Given two fuzzy sets $A, B \in I^X$, $A \subseteq B$ (or $A \leq B$) if $A(x) \leq B(x), \forall x \in X$.

Also, $A \setminus B = A \land B'$.

For further definitions regarding fuzzy sets, refer [1]

Definition 2.4 [3] A family $C$ of fuzzy subsets of $X$ is called a fuzzy convexity if

(i) $0, 1 \in C$

(ii) If $F \subseteq C$, then $\land F \in C$

(iii) If $F \subseteq C$ is non-empty and totally ordered by inclusion, then $\lor F \in C$.

The pair $(X, C)$ is called a fuzzy convex structure or a fuzzy convexity space. Members of $C$ are called fuzzy convex sets.

Note. $a$ denotes a constant function whose value at $x$ is $a, \forall x \in X$

Definition 2.5 [2] If $F$ is any fuzzy subset, then convex hull of $F$, is defined as

$$\text{Co}(F) = \land \{A \in C | F \subseteq A\}.$$
Definition 2.7 [2] Let \((X, C)\) be a fuzzy convex structure. A nonzero finite fuzzy subset \(F\) of \(X\) is exchange dependent (or \(E\)-dependent) if for each \(p_\alpha \in F\),

\[
\text{Co}(F \setminus p_\alpha) \leq \vee (\text{Co}(F \setminus a_\beta); a_\beta \in F \setminus p_\alpha; a \neq p).
\]

Otherwise, it is \(E\)-independent.

The exchange number of \(X\) is the smallest \(n\) such that each nonzero finite fuzzy subset \(F\) of \(X\), with cardinality of its support at least \(n + 1\), is exchange dependent. It is denoted by \(e(X)\) or by \(e\).

Definition 2.8 [4] Let \(\{(X_i, C_i); i \in I\}\) be a family of fuzzy convex structures. Let \(X = \prod_{i \in I} X_i\) be the product and let \(\pi_i : X \to X_i\) be the projection map. Then \(X\) can be equipped with the fuzzy convexity \(C\) generated by the fuzzy convex sets of the form

\[
\{\pi_i^{-1}(C_i); C_i \in C_i; i \in I\}.
\]

Then \(C\) is called the product fuzzy convexity on \(X\) and \((X, C)\) is called the product fuzzy convexity space.

Definition 2.9 [4] In a product space, the polytopes are of the product type i.e., for each finite fuzzy subset \(F\) of the product, \(\text{Co}(F) = \prod_{i \in I} \text{Co}(\pi_i(F))\).

Theorem 2.10 [2] Let \((X, C)\) be a fuzzy convexity space and let \(n < \infty\). Then \(c \leq n\) iff for each nonzero fuzzy subset \(A \subseteq X\) and \(p_\alpha \in \text{Co}(A)\) there is a fuzzy subset \(F\) of \(A\) with cardinality of its support at most \(n\) and having \(p_\alpha \in \text{Co}(F)\).

Theorem 2.11 [4] Let \((X_1, C_1), (X_2, C_2)\) be two fuzzy convexity spaces and \(X = X_1 \times X_2\), be the fuzzy convex product space. Suppose \(F_i \subseteq X_i\) be a fuzzy set with \(\#\text{Supp}(F_i) > 1\) for \(i = 1, 2\). Let \(p_i \in \text{Supp}(F_i)\) such that

\[
\text{Co}(F_i) \not\subseteq \vee \{\text{Co}(F_i \setminus q_\alpha); q_\alpha \in F_i; q \neq p_i\} \text{ for } i = 1, 2.
\]

Then the fuzzy subset \(F = [(p_1) \times (F_2 \setminus p_2)] \cup [(F_1 \setminus p_1) \times \{p_2\}]\) of \(X\) is \(C\)-independent.

3 Fuzzy convex invariants in a fuzzy convex product space

In this section, we establish some relationships between fuzzy convex invariants in a fuzzy convex product space with two factors.
Theorem 3.1 Let \((X_1,C_1), (X_2,C_2)\) be two fuzzy convexity spaces and \(X = X_1 \times X_2\), be the fuzzy convex product space. Let \(c_i\) and \(e_i\) be the Carathéodory number and exchange number of \(X_i\) for \(i = 1,2\). If \(c_i \geq e_i\) for \(i = 1,2\) then \(c = c_1 + c_2 - 2\) where \(c\) is the Carathéodory number of the product space \(X\).

Proof
Given that \(c_i \geq e_i\) for \(i = 1,2\). Then by definition, for each \(i\), there is a \(C\)-independent fuzzy set \(F_i \subseteq X_i\), with cardinality of its support as \(c_i\) and having a point \(p_i \in \text{Supp}(F_i)\) such that

\[
\text{Co}(F_i) \not\subseteq \bigvee \{\text{Co}(F_i \setminus q_\alpha); q_\alpha \in F_i; q \neq p_i\}.
\]

Then by Theorem 2.11, the fuzzy subset

\[
F = [(p_1 \times (F_2 \setminus p_2)) \vee ((F_1 \setminus p_1) \times \{p_2\})
\]

of \(X\) is \(C\)-independent. As the cardinality of support of \(F\) is at least \(c_1 + c_2 - 2\),

\[
c \geq c_1 + c_2 - 2. \quad (1)
\]

Let \(A\) be a fuzzy subset of \(X\) and \(p_\alpha \in \text{Co}(A)\).

Let \(\pi_i : X \to X_i; i = 1,2\) denotes the \(i^{\text{th}}\) projection. Then \(\pi_1(A)\) and \(\pi_2(A)\) are the projections of \(A\) on \(X_1\) and \(X_2\) respectively. Given that \(c_i\) is the Carathéodory number of \(X_i\) for \(i = 1,2\). Hence by Theorem 2.10, for each \(\pi_i(A) \subseteq X_i\) and \(\pi_i(p_\alpha) \in \text{Co}(\pi_i(A))\), there is a fuzzy subset \(F_i\) of \(A\) with cardinality of its support at most \(c_i\) such that \(\pi_i(p_\alpha) \in \text{Co}(\pi_i(F_i))\) for \(i = 1,2\).

But

\[
p_\alpha \in \text{Co}(A) \Rightarrow p_\alpha \in \text{Co}(\pi_1F_1) \times \text{Co}(\pi_2F_2) \quad \text{where}
\]

\[
\text{Co}(\pi_1F_1) \times \text{Co}(\pi_2F_2) \subseteq \text{Co}(\pi_1(F_1 \lor F_2)) \times \text{Co}(\pi_2(F_1 \lor F_2))
\]

\[
= \text{Co}(F_1 \lor F_2)
\]

Thus

\[
p_\alpha \in \text{Co}(F_1 \lor F_2).
\]

If \#\text{Supp}(F_i) < c_i for \(i = 1,2\) then

\[
\#\text{Supp}(F_1 \lor F_2) \leq c_1 + c_2 - 2.
\]

If \#\text{Supp}(F_i) = c_i for \(i = 1,2\) then take a fuzzy point \(a_{1i_1} \in F_i \setminus F_2\).

Since \#\text{Supp}(F_2 \lor a_{1i_1}) > c_2 \geq e_2\), the fuzzy subset \(F_2 \lor a_{1i_1}\) is E-dependent. Then by definition,

\[
\text{Co}(F_2) \leq \bigvee \{\text{Co}(F_2 \lor a_{1i_1}) \setminus b_{1i_1}; b_{1i_1} \in F_2\}
\]

i.e., \(\text{Co}(F_2) \leq \bigvee \text{Co}(F_2^*\) where \(F_2^* = F_2 \lor a_{1i_1} \setminus b_{1i_1}\).
Similarly, take a fuzzy point \( a_{2_{a_2}} \in F_2 \backslash F_1 \).

As \( \# \text{Supp} \left( F_1 \lor a_{2_{a_2}} \right) > c_1 \geq e_1 \), the fuzzy subset \( F_1 \lor a_{2_{a_2}} \) is E-dependent. Then by definition,

\[
\text{Co} ( F_1 ) \leq \lor \left\{ \text{Co} \left( \left[ F_1 \lor a_{2_{a_2}} \right] \backslash b_{2_{a_2}} \right) ; b_{2_{a_2}} \in F_1 \right\}
\]
i.e., \( \text{Co} ( F_1 ) \leq \lor \{ \text{Co} ( F_1^* ) \) where \( F_1^* = F_1 \lor a_{2_{a_2}} \backslash b_{2_{a_2}} \).

Since \( \pi_i ( p_\alpha ) \in \text{Co} ( \pi_i ( F_i^* ) ) \leq \text{Co} ( \pi_i ( F_i^* ) ) ; i = 1, 2 \)

\( p_\alpha \in \text{Co} ( F_1^* \lor F_2^* ) \)

where cardinality of support of \( F_1^* \lor F_2^* \) is \( c_1 + c_2 - 2 \). Thus,

\[ c \leq c_1 + c_2 - 2. \quad (2) \]

Combining (1) and (2),

\[ c = c_1 + c_2 - 2. \]

**Theorem 3.2** Let \( ( X_1, C_1 ), ( X_2, C_2 ) \) be two fuzzy convexity spaces and \( X = X_1 \times X_2 \), be the fuzzy convex product space. Suppose \( F_i \subseteq X_i \) be a fuzzy set with \( \# \text{Supp} ( F_i ) > 1 \) for \( i = 1, 2 \). Let \( p_i \in \text{Supp} ( F_i ) \) such that

\[ \text{Co} ( F_i ) \not\subseteq \lor \{ \text{Co} ( F_i \backslash q_\alpha ) ; q_\alpha \in F_i ; q \neq p_i \} \) for \( i = 1, 2 \).

Then the fuzzy subset \( F = ( F_1 \times \{ p_2 \} ) \lor ( \{ p_1 \} \times F_2 ) \) of \( X \) is \( E \)-independent.

**Proof**

For \( i = 1, 2 \), consider a fuzzy point \( x_{i_{\lambda_i}} \) in \( \text{Co} ( F_i ) \) but not in the sets

\[ \text{Co} ( F_i \backslash q_\alpha ) ; q_\alpha \in F_i ; q \neq p_i. \]

i.e., \( x_{i_{\lambda_i}} \in \text{Co} ( F_i ) \backslash \lor \{ \text{Co} ( F_i \backslash q_\alpha ) ; q_\alpha \in F_i ; q \neq p_i \} \) for \( i = 1, 2 \). \quad (3)

Let \( x_{\lambda} = ( x_{1_{\lambda_1}}, x_{2_{\lambda_2}} ) \) and \( p = ( p_1, p_2 ) \). As \( p_i \in \text{Supp} ( F_i ) \), \( p = ( p_1, p_2 ) \in F \).

Since \( \# \text{Supp} ( F_i ) > 1 \), \( \pi_i ( F \backslash p ) = F_i \) for \( i = 1, 2 \). Also by the product formula for polytopes, we have,

\[ \text{Co} ( F \backslash p ) = \text{Co} ( \pi_1 \{ F \backslash p \} ) \times \text{Co} ( \pi_2 \{ F \backslash p \} ) . \]

Hence \( x_{\lambda} = ( x_{1_{\lambda_1}}, x_{2_{\lambda_2}} ) \in \text{Co} ( F \backslash p ) . \)

Let \( a_{\beta} \) be a fuzzy point in \( F \) distinct from \( p \). Then either \( a_{\beta} = ( q_\beta, p_2 ) \) with \( q \neq p_1 \) or \( a_{\beta} = ( p_1, q_\beta ) \) with \( q \neq p_2 \). If \( a_{\beta} = ( q_\beta, p_2 ) \) then \( \pi_1 ( F \backslash a_{\beta} ) = F_1 \backslash q_\beta \) and \( \pi_2 ( F \backslash a_{\beta} ) = F_2 \). But by (1),

\[ x_{1_{\lambda_1}} \notin \text{Co} ( F_1 \backslash q_\alpha ) ; q_\alpha \in F_1 ; q \neq p_1 \text{ i.e., } x_{1_{\lambda_1}} \notin \text{Co} ( \pi_1 ( F \backslash a_{\beta} ) ). \]

Similarly, if \( a_{\beta} = ( p_1, q_\beta ) \) with \( q \neq p_2 \), we can show that \( x_{2_{\lambda_2}} \notin \text{Co} ( \pi_2 ( F \backslash a_{\beta} ) ) \).

Hence in either case, \( x_{\lambda} \notin \lor \{ \text{Co} ( F \backslash a_{\beta} ) ; a_{\beta} \in F \} \) where \( x_{\lambda} \in \text{Co} ( F \backslash p ) . \)

Therefore,

\[ \text{Co} ( F \backslash p ) \subseteq \lor \{ \text{Co} ( F \backslash a_{\beta} ) ; a_{\beta} \in F ; a \neq p \} . \]

Hence, by definition, \( F \) is \( E \)-independent.
Theorem 3.3 Let \((X_1, C_1)\), \((X_2, C_2)\) be two fuzzy convexity spaces and \(X = X_1 \times X_2\), be the fuzzy convex product space. Let \(c_i\) and \(e_i\) be the Caratheodory number and exchange number of \(X_i\) for \(i = 1, 2\) and \(e\) be the exchange number of the product space \(X\). Then the following assertions hold.

i) If \(c_i < e_i\) for \(i = 1, 2\) then \(e \geq e_1 + e_2 - 1\).

ii) If \(c_1 < e_1 \text{ and } c_2 \geq e_2\) then \(e \geq c_1 + e_2 - 1\).

iii) If \(c_1 \geq e_1 \text{ and } c_2 < e_2\) then \(e \geq c_1 + e_2 - 1\).

iv) If \(c_i \geq e_i\) for \(i = 1, 2\) then \(e \geq c_1 + c_2 - 1\).

Proof

i) Since \(e_i > c_i\) for \(i = 1, 2\) for each \(i\), there is a nonexchangeable fuzzy set \(F_i \subseteq X_i\) with cardinality of its support at least \(e_i\). In other words, \(F_i\) for \(i = 1, 2\) is \(E\) - independent. Then, by definition, for each \(i\), there is a point \(p_i \in \text{Supp}(F_i)\) such that

\[
\text{Co}(F_i \setminus p_i) \not\subseteq \bigvee \{\text{Co}(F_i \setminus q_\alpha); q_\alpha \in F_i; q \neq p_i\}.
\]

Hence

\[
\text{Co}(F_i) \not\subseteq \bigvee \{\text{Co}(F_i \setminus q_\alpha); q_\alpha \in F_i; q \neq p_i\} \text{ for } i = 1, 2.
\]

Then by Theorem 3.2, the fuzzy subset

\[
F = (F_1 \times \{p_2\}) \vee (\{p_1\} \times F_2)
\]

of \(X\) is \(E\) - independent.

Since the cardinality of support of \(F\) is at least \(e_1 + e_2 - 1\), we have

\[
e \geq e_1 + e_2 - 1.
\]

ii) Since \(e_1 > c_1\) as above, there is a nonexchangeable fuzzy set \(F_1 \subseteq X_1\) with cardinality of its support at least \(e_1\), i.e., \(F_1\) is \(E\) - independent.

Then, by definition, there is a point \(p_1 \in \text{Supp}(F_1)\) such that

\[
\text{Co}(F_1 \setminus p_1) \not\subseteq \bigvee \{\text{Co}(F_1 \setminus q_\alpha); q_\alpha \in F_1; q \neq p_1\}.
\]

Hence

\[
\text{Co}(F_1) \not\subseteq \bigvee \{\text{Co}(F_1 \setminus q_\alpha); q_\alpha \in F_1; q \neq p_1\}.
\]

Since \(c_2 \geq e_2\), there is a \(C\) - independent fuzzy set \(F_2 \subseteq X_2\) with cardinality of its support as \(c_2\) and having a point \(p_2 \in \text{Supp}(F_2)\) such that

\[
\text{Co}(F_2) \not\subseteq \bigvee \{\text{Co}(F_2 \setminus q_\alpha); q_\alpha \in F_2; q \neq p_2\}.
\]

Then by Theorem 3.2, the fuzzy subset

\[
F = (F_1 \times \{p_2\}) \vee (\{p_1\} \times F_2)
\]
of $X$ is $E$- independent. As the cardinality of support of $F$ is at least $e_1 + c_2 - 1$, 

$$e \geq e_1 + c_2 - 1. \quad (5)$$

iii) Similarly, it can be proved that 

$$e \geq c_1 + e_2 - 1. \quad (6)$$

iv) Since $c_i \geq c_i$, for $i = 1, 2$, for each $i$, there is a $C$- independent fuzzy set $F_i \subseteq X_i$, with cardinality of its support as $c_i$ and having a point $p_i \in \text{Supp}(F_i)$ such that 

$$\text{Co}(F_i) \nsubseteq \bigvee \{\text{Co}(F_i \setminus q); q \in F_i; q \neq p_i\}.$$ 

Then by Theorem 3.2, the fuzzy subset 

$$F = (F_1 \times \{p_2\}) \lor (\{p_1\} \times F_2)$$ 

of $X$ is $E$- independent. As the cardinality of support of $F$ is at least $c_1 + c_2 - 1$, 

$$e \geq c_1 + c_2 - 1. \quad (7)$$ 

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**References**


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