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On Some Fuzzy Convex Invariants in a Fuzzy Convex Product Space

Lissy Jacob

Department of Mathematics
Bharata Mata College, Thrikkakara, Kerala, India, PIN-682021
lisfrancy@rediffmail.com

M. V. Rosa

Department of Mathematics
Bharata Mata College, Thrikkakara,Kerala, India, PIN-682021
mvrbmc@gmail.com

Abstract

J. Eckhoff introduced the concept of convex product space in 1968. The classical convex invariants namely Helly number, Caratheodory number, Radon number and exchange number for the convex product space are determined and studied by G.Sierksma [5] in 1976. In [4],the authors studied some of these classical convex invariants in the fuzzy context and arrived at certain results. In this paper, the exchange number of a fuzzy convex product space with two factors is discussed in various situations.

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1 Introduction

This paper is a continuation of the study of fuzzy convex invariants in a fuzzy convex product space with two factors [4]. In [4] , the Caratheodory number of a fuzzy convex product space with two factors is derived in connection with the exchange number of the factor spaces. In this paper, the exchange number of a convex product space with two factors is studied in the fuzzy context and some relations are established.

2 Basic definitions and properties

Definition 2.1 [1] *A fuzzy subset A on a nonempty set X is a function from X to $I = [0, 1]$. The set of all fuzzy subsets of X is denoted by I^X.*

A fuzzy point a_α is a fuzzy subset defined as

$$a_\alpha(y) = \begin{cases} \alpha > 0, \text{if } y = a \\ 0, \text{ otherwise} \end{cases}$$

Definition 2.2 [1] *Support of a fuzzy subset A is $Supp\,(A) = \{x \in X | A\,(x) > 0\}$. A fuzzy subset is said to be finite if its support is finite.*

Definition 2.3 [1] *Given two fuzzy sets $A, B \in I^X, A \subseteq B\,(or\,A \leq B)$ if $A\,(x) \leq B\,(x)\,, \forall x \in X$.*

Also, $A \backslash B = A \wedge B'$.

For further definitions regarding fuzzy sets, refer [1]

Definition 2.4 [3] *A family $\mathcal{C}$ of fuzzy subsets of X is called a fuzzy convexity if*
(i) $\underline{0}$, $\underline{1} \in \mathcal{C}$
(ii) $If\ F \subseteq \mathcal{C}$, then $\wedge F \in \mathcal{C}$
(iii) If $F \subseteq \mathcal{C}$ is non-empty and totally ordered by inclusion, then $\vee F \in \mathcal{C}$.

The pair $(X, \mathcal{C})$ is called a fuzzy convex structure or a fuzzy convexity space. Members of $\mathcal{C}$ are called fuzzy convex sets.

Note. $\underline{a}$ denotes a constant function whose value at x is $a, \forall x \in X$

Definition 2.5 [2] *If F is any fuzzy subset, then convex hull of F, is defined as*

$$Co\,(F) = \wedge \{A \in \mathcal{C} | F \subseteq A\}\,.$$

Definition 2.6 [2] *Let $(X, \mathcal{C})$ be a fuzzy convex structure. A nonzero finite fuzzy subset F of X is Caratheodory dependent (or C-dependent) provided*

$$Co\,(F) \leq \bigvee_{a_\alpha \in F} Co\,(F \backslash a_\alpha)$$

Otherwise, it is C-independent.

The Caratheodory number of X is the smallest 'n' such that each nonzero finite fuzzy subset F of X , with cardinality of its support at least '$n + 1$', is Caratheodory dependent. It is denoted by $c(X)$ or by c.

Definition 2.7 [2] *Let $(X, \mathcal{C})$ be a fuzzy convex structure. A nonzero finite fuzzy subset F of X is exchange dependent (or E-dependent) if for each $p_\alpha \in F$,*

$$Co\,(F \backslash p_\alpha) \leq \vee \left(Co\,(F \backslash a_\beta)\,; a_\beta \in F \backslash p_\alpha; a \neq p\right).$$

Otherwise, it is E-independent.

The exchange number of X is the smallest 'n' such that each nonzero finite fuzzy subset F of X, with cardinality of its support at least '$n+1$', is exchange dependent. It is denoted by $e(X)$ or by e.

Definition 2.8 [4] *Let $\{(X_i, \mathcal{C}_i)\,; i \in I\}$ be a family of fuzzy convex structures. Let $X = \prod_{i \in I} X_i$ be the product and let $\pi_i : X \to X_i$ be the projection map. Then X can be equipped with the fuzzy convexity $\mathcal{C}$ generated by the fuzzy convex sets of the form*

$$\left\{\pi_i^{-1}\left(\mathrm{C}_i\right)/\mathrm{C}_i \in \mathcal{C}_i; i \in I\right\}.$$

Then $\mathcal{C}$ is called the product fuzzy convexity on X and $(X, \mathcal{C})$ is called the product fuzzy convexity space.

Definition 2.9 [4] *In a product space, the polytopes are of the product type i.e., for each finite fuzzy subset F of the product,$Co\,(F) = \prod_{i \in I} Co\,(\pi_i\,(F))$.*

Theorem 2.10 [2] *Let $(X, \mathcal{C})$ be a fuzzy convexity space and let $n < \infty$. Then $c \leq n$ iff for each nonzero fuzzy subset $A \subseteq X$ and $p_\alpha \in Co\,(A)$ there is a fuzzy subset F of A with cardinality of its support at most n and having $p_\alpha \in Co\,(F)$.*

Theorem 2.11 [4] *Let $(X_1, \mathcal{C}_1)$, $(X_2, \mathcal{C}_2)$ be two fuzzy convexity spaces and $X = X_1 \times X_2$, be the fuzzy convex product space. Suppose $F_i \subseteq X_i$ be a fuzzy set with $\#Supp(F_i) > 1$ for $i = 1, 2$. Let $p_i \in Supp(F_i)$ such that*

$$Co(F_i) \nleq \vee \left\{Co(F_i \backslash q_\alpha); q_\alpha \in F_i; q \neq p_i\right\} for\ i = 1, 2.$$

Then the fuzzy subset $F = [\{p_1\} \times (F_2 \setminus p_2)] \vee [(F_1 \setminus p_1) \times \{p_2\}]$ of X is C- independent.

3 Fuzzy convex invariants in a fuzzy convex product space

In this section, we establish some relationships between fuzzy convex invariants in a fuzzy convex product space with two factors.

Theorem 3.1 *Let $(X_1, \mathcal{C}_1)$, $(X_2, \mathcal{C}_2)$ be two fuzzy convexity spaces and $X = X_1 \times X_2$, be the fuzzy convex product space. Let c_i and e_i be the Caratheodory number and exchange number of X_i for $i = 1, 2$. If $c_i \geq e_i$ for $i = 1, 2$ then $c = c_1 + c_2 - 2$ where c is the Caratheodory number of the product space X .*

Proof

Given that $c_i \geq e_i$ for $i = 1, 2$.Then by definition, for each i, there is a C- independent fuzzy set $F_i \subseteq X_i$, with cardinality of its support as c_i and having a point $p_i \in Supp(F_i)$ such that

$$Co(F_i) \nleq \vee\{Co(F_i \backslash q_\alpha); q_\alpha \in F_i; q \neq p_i\}.$$

Then by Theorem 2.11, the fuzzy subset

$$F = [\{p_1\} \times (F_2 \setminus p_2)] \vee [(F_1 \setminus p_1) \times \{p_2\}]$$

of X is C- independent. As the cardinality of support of F is at least c_1+c_2-2,

$$c \geq c_1 + c_2 - 2. \tag{1}$$

Let A be a fuzzy subset of X and $p_\alpha \in Co\,(A)$.
Let $\pi_i : X \to X_i; i = 1, 2$ denotes the i^{th} projection. Then $\pi_1\,(A)$ and $\pi_2\,(A)$ are the projections of A on X_1 and X_2 respectively. Given that c_i is the Caratheodory number of X_i for $i = 1, 2$. Hence by Theorem 2.10, for each $\pi_i\,(A) \subseteq X_i$ and $\pi_i\,(p_\alpha) \in Co\,(\pi_i\,(A))$, there is a fuzzy subset F_i of A with cardinality of its support at most c_i such that $\pi_i\,(p_\alpha) \in Co\,(\pi_i\,(F_i))$ for $i = 1, 2$. But

$$p_\alpha \in Co\,(A) \Rightarrow p_\alpha \in Co\,(\pi_1 F_1) \times Co\,(\pi_2 F_2) \;\; where$$

$$\begin{aligned} Co\,(\pi_1 F_1) \times Co\,(\pi_2 F_2) &\subseteq Co\,(\pi_1\,(F_1 \vee F_2)) \times Co\,(\pi_2\,(F_1 \vee F_2)) \\ &= Co\,(F_1 \vee F_2) \end{aligned}$$

Thus

$$p_\alpha \in Co\,(F_1 \vee F_2)\,.$$

If $\#Supp\,(F_i) < c_i$ for $i = 1, 2$ then

$$\#Supp\,(F_1 \vee F_2) \leq c_1 + c_2 - 2.$$

If $\#Supp\,(F_i) = c_i$ for $i = 1, 2$ then take a fuzzy point $a_{1_{\alpha_1}} \in F_1 \backslash F_2$. Since $\#Supp\left(F_2 \vee a_{1_{\alpha_1}}\right) > c_2 \geq e_2$, the fuzzy subset $F_2 \vee a_{1_{\alpha_1}}$ is E-dependent. Then by definition,

$$Co\,(F_2) \leq \vee\left\{Co\left(\,[F_2 \vee a_{1_{\alpha_1}}\,]\backslash b_{1_{\beta_1}}\right); b_{1_{\beta_1}} \in F_2\right\}$$

i.e., $Co\,(F_2) \leq \vee Co\,(F_2^*)$ where $F_2^* = F_2 \vee a_{1_{\alpha_1}} \backslash b_{1_{\beta_1}}$.

Similarly, take a fuzzy point $a_{2_{\alpha_2}} \in F_2 \backslash F_1$.
As $\#Supp\left(F_1 \vee a_{2_{\alpha_2}}\right) > c_1 \geq e_1$, the fuzzy subset $F_1 \vee a_{2_{\alpha_2}}$ is E-dependent. Then by definition,

$$Co\,(F_1) \leq \vee \left\{ Co\left(\; [F_1 \vee a_{2_{\alpha_2}} \;] \backslash b_{2_{\beta_2}} \right); b_{2_{\beta_2}} \in F_1 \right\}$$

i.e., $Co\,(F_1) \leq \vee Co\,(F_1^*)$ where $F_1^* = F_1 \vee a_{2_{\alpha_2}} \backslash b_{2_{\beta_2}}$.

Since $\pi_i\,(p_\alpha) \in Co\,(\pi_i\,(F_i)) \leq Co\,(\pi_i\,(F_i^*))\,; i = 1, 2$

$$p_\alpha \in Co\,(F_1^* \vee F_2^*)$$

where cardinality of support of $F_1^* \vee F_2^*$ is $c_1 + c_2 - 2$. Thus,

$$c \leq c_1 + c_2 - 2. \tag{2}$$

Combining (1) and (2),

$$c = c_1 + c_2 - 2.$$

Theorem 3.2 *Let $(X_1, \mathcal{C}_1)$, $(X_2, \mathcal{C}_2)$ be two fuzzy convexity spaces and $X = X_1 \times X_2$, be the fuzzy convex product space. Suppose $F_i \subseteq X_i$ be a fuzzy set with $\#Supp(F_i) > 1$ for $i = 1, 2$. Let $p_i \in Supp(F_i)$ such that*

$$Co(F_i) \nleq \vee \{Co(F_i \backslash q_\alpha); q_\alpha \in F_i; q \neq p_i\} for \;\; i = 1, 2.$$

Then the fuzzy subset $F = (F_1 \times \{p_2\}) \vee (\{p_1\} \times F_2)$ of X is E- independent.

Proof

For $i = 1, 2$, consider a fuzzy point $x_{i_{\lambda_i}}$ in $Co(F_i)$ but not in the sets

$$Co(F_i \backslash q_\alpha); q_\alpha \in F_i; q \neq p_i.$$

$$i.e., x_{i_{\lambda_i}} \in Co(F_i) \setminus \vee \{Co(F_i \backslash q_\alpha); q_\alpha \in F_i; q \neq p_i\} \text{for i} = 1, 2. \tag{3}$$

Let $x_\lambda = (x_{1_{\lambda_1}}, x_{2_{\lambda_2}})$ and $p = (p_1, p_2)$. As $p_i \in Supp(F_i)$, $p = (p_1, p_2) \in F$. Since $\#Supp(F_i) > 1$, $\pi_i(F \backslash p) = F_i$ for $i = 1, 2$. Also by the product formula for polytopes, we have,

$$Co\,(F \backslash p) = Co\,(\pi_1\,\{F \backslash p\}) \times Co\,(\pi_2\,\{F \backslash p\})\,.$$

Hence $x_\lambda = (x_{1_{\lambda_1}}, x_{2_{\lambda_2}}) \in Co(F \backslash p)$.

Let a_β be a fuzzy point in F distinct from p . Then either $a_\beta = (q_\beta, p_2)$ with $q \neq p_1$ or $a_\beta = (p_1, q_\beta)$ with $q \neq p_2$. If $a_\beta = (q_\beta, p_2)$ then $\pi_1(F \setminus a_\beta) = F_1 \setminus q_\beta$ and $\pi_2(F \setminus a_\beta) = F_2$. But by (1),

$$x_{1_{\lambda_1}} \notin Co(F_1 \backslash q_\alpha); q_\alpha \in F_1; q \neq p_1 . i.e., x_{1_{\lambda_1}} \notin Co(\pi_1(F \backslash a_\beta)).$$

Similarly, if $a_\beta = (p_1, q_\beta)$ with $q \neq p_2$, we can show that $x_{2_{\lambda_2}} \notin Co(\pi_2(F \backslash a_\beta))$. Hence in either case, $x_\lambda \notin \vee \{Co\,(F \backslash a_\beta)\,; a_\beta \in F\}\,;$ where $x_\lambda \in Co(F \backslash p)$. Therefore,

$$Co\,(F \backslash p) \nleq \vee \{Co\,(F \backslash a_\beta)\,; a_\beta \in F; a \neq p\}\,.$$

Hence, by definition, F is E- independent.

Theorem 3.3 *Let $(X_1, \mathcal{C}_1)$, $(X_2, \mathcal{C}_2)$ be two fuzzy convexity spaces and $X = X_1 \times X_2$, be the fuzzy convex product space. Let c_i and e_i be the Caratheodory number and exchange number of X_i for $i = 1, 2$ and e be the exchange number of the product space X. Then the following assertions hold.*

i) If $c_i < e_i fori = 1, 2$ then $e \geq e_1 + e_2 - 1$

ii) If $c_1 < e_1 andc_2 \geq e_2$ then $e \geq e_1 + c_2 - 1$

iii) If $c_1 \geq e_1 andc_2 < e_2$ then $e \geq c_1 + e_2 - 1$

iv) If $c_i \geq e_i fori = 1, 2$ then $e \geq c_1 + c_2 - 1$.

Proof

i) Since $e_i > c_i$ for $i = 1, 2$ for each i, there is a nonexchangeable fuzzy set $F_i \subseteq X_i$ with cardinality of its support at least e_i. In other words , F_i; for $i = 1, 2$ is E - independent. Then, by definition, for each i , there is a point $p_i \in Supp(F_i)$ such that

$$Co(F_i \setminus p_i) \nleq \vee\{Co(F_i \backslash q_\alpha); q_\alpha \in F_i; q \neq p_i\}.$$

Hence

$$Co(F_i) \nleq \vee\{Co(F_i \backslash q_\alpha); q_\alpha \in F_i; q \neq p_i\} \text{ for } i = 1, 2.$$

Then by Theorem 3.2, the fuzzy subset

$$F = (F_1 \times \{p_2\}) \vee (\{p_1\} \times F_2)$$

of X is E- independent.

Since the cardinality of support of F is at least $e_1 + e_2 - 1$, we have

$$e \geq e_1 + e_2 - 1. \tag{4}$$

ii) Since $e_1 > c_1$,as above, there is a nonexchangeable fuzzy set $F_1 \subseteq X_1$ with cardinality of its support at least e_1. i.e., F_1 is E - independent.

Then, by definition, there is a point $p_1 \in Supp(F_1)$ such that

$$Co(F_1 \setminus p_1) \nleq \vee\{Co(F_1 \backslash q_\alpha); q_\alpha \in F_1; q \neq p_1\}.$$

Hence

$$Co(F_1) \nleq \vee\{Co(F_1 \backslash q_\alpha); q_\alpha \in F_1; q \neq p_1\}.$$

Since $c_2 \geq e_2$, there is a C- independent fuzzy set $F_2 \subseteq X_2$, with cardinality of its support as c_2 and having a point $p_2 \in Supp(F_2)$ such that

$$Co(F_2) \nleq \bigvee\{Co(F_2 \backslash q_\alpha); q_\alpha \in F_2; q \neq p_2\}.$$

Then by Theorem 3.2, the fuzzy subset

$$F = (F_1 \times \{p_2\}) \vee (\{p_1\} \times F_2)$$

of X is E- independent. As the cardinality of support of F is at least e_1+c_2-1,

$$e \geq e_1 + c_2 - 1. \tag{5}$$

iii) Similarly, it can be proved that

$$e \geq c_1 + e_2 - 1. \tag{6}$$

iv) Since $c_i \geq e_i$, for $i = 1, 2$, for each i , there is a C- independent fuzzy set $F_i \subseteq X_i$, with cardinality of its support as c_i and having a point $p_i \in Supp(F_i)$ such that

$$Co(F_i) \nleq \vee\{Co(F_i \backslash q_\alpha); q_\alpha \in F_i; q \neq p_i\}.$$

Then by Theorem 3.2, the fuzzy subset

$$F = (F_1 \times \{p_2\}) \vee (\{p_1\} \times F_2)$$

of X is E- independent. As the cardinality of support of F is at least c_1+c_2-1,

$$e \geq c_1 + c_2 - 1. \tag{7}$$

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