On the Sublinear Wang’s Premium Principle

Gianni Bosi

Dipartimento di Scienze Economiche, Aziendali
Matematiche e Statistiche, Università di Trieste
Piazzale Europa 1, 34127 Trieste, Italy
GIANNI.BOSI@econ.units.it

Magali E. Zuanon

Dipartimento di Metodi Quantitativi
Università degli Studi di Brescia
Contrada Santa Chiara 50, 25122 Brescia, Italy

Abstract

The so called “Wang’s premium” is the principle of premium calculation expressed by means of the Choquet integral with respect to a (concave) distorted probability. In this paper we present a simple axiomatization of a sublinear Wang’s premium on the space of all the nonnegative bounded risks on a common probability space.

Mathematics Subject Classification: 91B30; 28B20

Keywords: Wang’s premium principle; Choquet integral; sublinear functional

1 Introduction

In this paper a premium functional \( P \) is referred to as a functional from the space \( L^\infty_+(\Omega, \mathcal{F}, \mathcal{P}) \) of all the nonnegative risks (i.e., nonnegative bounded random variables) on a common probability space \( (\Omega, \mathcal{F}, \mathcal{P}) \) into \( \mathbb{R}^+ \). For a risk \( X \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P}) \), \( X(\omega) \) is obviously interpreted as the loss that the insurance company incurs if the outcome is \( \omega \in \Omega \).

Several authors were concerned with the representability of an (extended) real-valued function on a space of risks by means of a Choquet integral (see e.g. Chateauneuf [4], Young [15], Wang, Young and Panjer [11], Wirch and Hardy [12], Wu and Wang [13], Dhaene et al. [6], Wu and Wang [14] and Bosi and Zuanon [3]). More recently, very deep results in this direction have been
presented by Song and Yan [9, 10].

There exists a representation of this kind if there is a probability distortion $g$ such that, for every $X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$,

$$
\mathbb{P}(X) = \int X \, dg \circ \mathcal{P} = \int_0^{+\infty} g(\mathcal{P}(\{\omega \in \Omega : X(\omega) > t\})) \, dt.
$$

In this case, $\mathbb{P}$ is said to be a Wang's premium principle in the actuarial literature (see e.g. [13]).

We characterize the representability of a sublinear premium functional $\mathbb{P}$ on $L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ as the Choquet integral with respect to a concave probability distortion. To this aim, we use both results presented by Parker [8] and by Song and Yan [10]. In particular we show that a monotone sublinear premium functional is a Wang's premium if and only if it satisfies the assumption of no unjustified loading and it is widely translation invariant and monotone with respect to stop-loss order.

## 2 Notation and preliminaries

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, and denote by $\mathbb{1}_F$ the indicator function of any subset $F$ of $\Omega$. In the sequel, we shall denote by $\mathbb{R}^+$ the set of all nonnegative real numbers. For the sake of brevity, for every $c \in \mathbb{R}^+$ we identify $c$ with the constant function $c \mathbb{1}_\Omega$.

Denote by $L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ the space of all the nonnegative bounded random variables $X$ on $(\Omega, \mathcal{F}, \mathcal{P})$.

If for two $X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ we have that $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$, then we shall simply write $X \leq Y$. We have that $\leq$ is a preorder (i.e., $\leq$ is a reflexive and transitive binary relation).

Denote by $S_X(t) = 1 - F_X(t) = \mathbb{P}(\{\omega \in \Omega : X(\omega) > t\})$ the decumulative distribution function of any random variable $X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$.

A premium functional $\mathbb{P}$ is intended as a functional from the space $L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ into $\mathbb{R}^+$. A premium functional is said to be

1. **Monotone** if $\mathbb{P}(X) \leq \mathbb{P}(Y)$ for all $X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ such that $X \leq Y$;

2. **Monotone with respect to first order stochastic dominance** if $\mathbb{P}(X) \leq \mathbb{P}(Y)$ for all $X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ such that $S_X(t) \leq S_Y(t)$ for all $t \in \mathbb{R}^+$;

3. **Monotone with respect to stop-loss order** if $\mathbb{P}(X) \leq \mathbb{P}(Y)$ for all $X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ such that $\mathbb{E}[(X-\mathcal{P})_+] \leq \mathbb{E}[(Y-\mathcal{P})_+]$ for all $\mathcal{P} \in \mathbb{R}^+$;

4. **Translation Invariant** if $\mathbb{P}(X+c) = \mathbb{P}(X) + c$ for all $X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ and $c \in \mathbb{R}^+$;
5. **Widely Translation Invariant** if $\mathbb{P}(X + c \mathbb{1}_F) = \mathbb{P}(X) + c\mathbb{P}(\mathbb{1}_F)$ for all $X \in L^\infty_+(\Omega, \mathcal{F}, \mathbb{P})$, $c \in \mathbb{R}^+$ and $F \in \mathcal{F}$ such that $\{X > 0\} \subseteq F$;

6. **Comonotone Additive** if $\mathbb{P}(X + Y) = \mathbb{P}(X) + \mathbb{P}(Y)$ for all comonotone $X, Y \in L^\infty_+(\Omega, \mathcal{F}, \mathbb{P})$ (i.e., for all $X, Y \in L^\infty_+(\Omega, \mathcal{F}, \mathbb{P})$ such that $(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0$ for all $\omega_1, \omega_2 \in \Omega$);

7. **Sublinear** if $\mathbb{P}$ is positively homogeneous (i.e., $\mathbb{P}(\gamma X) = \gamma \mathbb{P}(X)$ for every $\gamma \in \mathbb{R}^+$ and $X \in L^\infty_+(\Omega, \mathcal{F}, \mathbb{P})$) and subadditive (i.e., $\mathbb{P}(X + Y) \leq \mathbb{P}(X) + \mathbb{P}(Y)$ for all $X, Y \in L^\infty_+(\Omega, \mathcal{F}, \mathbb{P})$);

It is clear that if a premium functional $\mathbb{P}$ on $L^\infty_+(\Omega, \mathcal{F}, \mathbb{P})$ is widely translation invariant then it is translation invariant.

A premium functional $\mathbb{P}$ is said to satisfy the axiom of **no unjustified loading** if $\mathbb{P}(\mathbb{1}_\Omega) = 1$.

In the next section necessary and sufficient conditions are presented on a premium functional $\mathbb{P}$ on $L^\infty_+(\Omega, \mathcal{F}, \mathbb{P})$ under which there exists a concave probability distortion $g$ (i.e., $g$ is a real-valued, nondecreasing, nonnegative and concave function on $[0, 1]$ such that $g(0) = 0$ and $g(1) = 1$) such that, for every $X \in L^\infty_+(\Omega, \mathcal{F}, \mathbb{P})$,

$$
\mathbb{P}(X) = \int Xdg \circ \mathbb{P} = \int_0^{+\infty} g(S_X(t))dt.
$$

In this case, $\mathbb{P}$ is the **Choquet integral** with respect to the distorted probability $\mu = g \circ \mathbb{P}$. If this happens, $\mathbb{P}$ is said to be a **sublinear Wang’s premium principle**. It should be noted that a sublinear Wang’s premium principle is a distortion risk measure (i.e., it is a coherent risk measure in the sense of Artzner et al. [1]), which in addition is expressed by means of a Choquet integral (see e.g. Balbás Garrido and Mayoral [2]).

### 3 Choquet integral representation of premium functionals

In the following theorem we characterize the representability of a premium functional $\mathbb{P}$ on $L^\infty_+(\Omega, \mathcal{F}, \mathbb{P})$ as a Wang’s premium principle with respect to a concave probability distortion. It is well known that if $\mathbb{P}$ is a Wang’s premium functional on $L^\infty_+(\Omega, \mathcal{F}, \mathbb{P})$ with respect to a concave probability distortion $g$, then $\mathbb{P}$ is subadditive and in particular sublinear (see Denneberg [5]).

We use the following nice result proved by Wirch and Hardy [12].

**Lemma 3.1** ([12], **Theorem 2.2**) Let $\mathbb{P}$ be a Wang’s premium functional on $L^\infty_+(\Omega, \mathcal{F}, \mathbb{P})$ with respect to a probability distortion $g$. If $\mathbb{P}$ is subadditive, then $g$ is concave.
Theorem 3.2 Let $\mathbb{P}$ be a sublinear premium functional on $L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ and assume that $\mathbb{P}$ satisfies the axiom of no unjustified loading. Then the following conditions are equivalent:

(i) There exists a concave probability distortion $g$ such that $\mathbb{P}(X) = \int X dg \circ \mathbb{P}$ for all $X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$;

(ii) The following conditions are verified:

(a) $\mathbb{P}$ is monotone with respect to stop loss order;

(b) for all $X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$, $c \in \mathbb{R}^+$ and $F \in \mathcal{F}$ such that $\{X > 0\} \subseteq F$:

$$\mathbb{P}(X + c1_F) \geq \mathbb{P}(X) + c\mathbb{P}(1_F).$$

Proof. (i) $\Rightarrow$ (ii). Assume that condition (i) holds. From Dhaene et al. [7, Theorem 8], we have that $\mathbb{P}$ is monotone with respect to stop-loss order. Since $\mathbb{P}$ is comonotone additive, we have that it is widely translation invariant and therefore it is clear that the above condition (b) holds.

(ii) $\Rightarrow$ (i). Now assume that condition (ii) is verified. Then we have that sublinearity of $\mathbb{P}$ and the above condition (b) immediately imply that $\mathbb{P}$ is widely translation invariant. Since $\mathbb{P}$ is positively homogeneous, widely translation invariant and monotone, it is comonotone additive (see Parker [8, Theorem 7]). Further, from Parker [8, Theorem 9], since $\mathbb{P}(1_\Omega) = 1$ we have that $\mathbb{P}$ is the Choquet integral with respect to the normalized set function $v$ on $\mathcal{F}$ defined by $v(F) = \mathbb{P}(1_F)$ for all $F \in \mathcal{F}$. Since in addition $\mathbb{P}$ is monotone with respect to first order stochastic dominance, from Song and Yan [10, Proposition 2.1 and Remark 2.2] there exists a distortion function $g$ such that $\mathbb{P}$ is the Choquet integral of $X$ with respect to the distorted probability $\mu = g \circ \mathcal{P}$. Since $\mathbb{P}$ is subadditive, we have that $g$ is concave from Lemma 3.1.

References


Received: October, 2011