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On the Sublinear Wang's Premium Principle

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Abstract

The so called "Wang's premium" is the principle of premium calculation expressed by means of the Choquet integral with respect to a (concave) distorted probability. In this paper we present a simple axiomatization of a sublinear Wang's premium on the space of all the nonnegative bounded risks on a common probability space.

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1 Introduction

In this paper a *premium functional* $\mathbb{P}$ is referred to as a functional from the space $L_{+}^{\infty}(\Omega, \mathcal{F}, \mathcal{P})$ of all the nonnegative *risks* (i.e., nonnegative bounded random variables) on a common probability space $(\Omega, \mathcal{F}, \mathcal{P})$ into $\mathbb{R}^{+}$. For a risk $X \in L_{+}^{\infty}(\Omega, \mathcal{F}, \mathcal{P})$, $X(\omega)$ is obviously interpreted as the loss that the insurance company incurs if the outcome is $\omega \in \Omega$.

Several authors were concerned with the representability of an (extended) real-valued function on a space of risks by means of a *Choquet integral* (see e.g. Chateauneuf [4], Young [15], Wang, Young and Panjer [11], Wirch and Hardy [12], Wu and Wang [13], Dhaene et al. [6], Wu and Wang [14] and Bosi and Zuanon [3]). More recently, very deep results in this direction have been presented by Song and Yan [9, 10].

There exists a representation of this kind if there is a *probability distortion* g such that, for every $X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$,

$$\mathbb{P}(X) = \int X dg \circ \mathcal{P} = \int_0^{+\infty} g(\mathcal{P}(\{\omega \in \Omega : X(\omega) > t\}))dt.$$

In this case, $\mathbb{P}$ is said to be a *Wang's premium principle* in the actuarial literature (see e.g. [13]).

We characterize the representability of a sublinear premium functional $\mathbb{P}$ on $L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ as the Choquet integral with respect to a concave probability distortion. To this aim, we use both results presented by Parker [8] and by Song and Yan [10]. In particular we show that a monotone sublinear premium functional is a Wang's premium if and only if it satisfies the assumption of no unjustified loading and it is widely translation invariant and monotone with respect to stop-loss order.

2 Notation and preliminaries

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a *probability space*, and denote by $\mathbb{1}_F$ the *indicator function* of any subset F of Ω. In the sequel, we shall denote by $\mathbb{R}^+$ the set of all nonnegative real numbers. For the sake of brevity, for every $c \in \mathbb{R}^+$ we identify c with the constant function $c\mathbb{1}_\Omega$.

Denote by $L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ the space of all the nonnegative bounded *random variables* on $(\Omega, \mathcal{F}, \mathcal{P})$.

If for two $X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ we have that $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$, then we shall simply write $X \leq Y$. We have that $\leq$ is a preorder (i.e., $\leq$ is a *reflexive* and *transitive* binary relation).

Denote by $S_X(t) = 1 - F_X(t) = \mathbb{P}(\{\omega \in \Omega : X(\omega) > t\})$ the *decumulative distribution function* of any random variable $X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$.

A *premium functional* $\mathbb{P}$ is intended as a functional from the space $L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ into $\mathbb{R}^+$. A premium functional is said to be

1. *Monotone* if $\mathbb{P}(X) \leq P(Y)$ for all $X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ such that $X \leq Y$;

2. *Monotone with respect to first order stochastic dominance* if $\mathbb{P}(X) \leq \mathbb{P}(Y)$ for all $X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ such that $S_X(t) \leq S_Y(t)$ for all $t \in \mathbb{R}^+$;

3. *Monotone with respect to stop-loss order* if $\mathbb{P}(X) \leq \mathbb{P}(Y)$ for all $X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}[(X - d)_+] \leq \mathbb{E}[(Y - d)_+]$ for all $d \in \mathbb{R}^+$;

4. *Translation Invariant* if $\mathbb{P}(X + c) = \mathbb{P}(X) + c$ for all $X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ and $c \in R^+$;

5. *Widely Translation Invariant* if $\mathbb{P}(X + c\mathbb{1}_F) = \mathbb{P}(X) + c\mathbb{P}(\mathbb{1}_F)$ for all $X \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$, $c \in \mathbb{R}^+$ and $F \in \mathcal{F}$ such that $\{X > 0\} \subseteq F$;

6. *Comonotone Additive* if $\mathbb{P}(X + Y) = \mathbb{P}(X) + \mathbb{P}(Y)$ for all comonotone $X, Y \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$ (i.e., for all $X, Y \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$ such that $(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0$ for all $\omega_1, \omega_2 \in \Omega$);

7. *Sublinear* if $\mathbb{P}$ is *positively homogeneous* (i.e., $\mathbb{P}(\gamma X) = \gamma\mathbb{P}(X)$ for every $\gamma \in \mathbb{R}^+$ and $X \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$) and *subadditive* (i.e., $\mathbb{P}(X+Y) \leq \mathbb{P}(X) + \mathbb{P}(Y)$ for all $X, Y \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$);

It is clear that if a premium functional $\mathbb{P}$ on $L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$ is widely translation invariant then it is translation invariant.

A premium functional $\mathbb{P}$ is said to satisfy the axiom of *no unjustified loading* if $\mathbb{P}(\mathbb{1}_\Omega) = 1$.

In the next section necessary and sufficient conditions are presented on a premium functional $\mathbb{P}$ on $L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$ under which there exists a *concave probability distortion* g (i.e., g is a real-valued, nondecreasing, nonnegative and concave function on $[0, 1]$ such that $g(0) = 0$ and $g(1) = 1$) such that, for every $X \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$,

$$\mathbb{P}(X) = \int X dg \circ \mathbb{P} = \int_0^{+\infty} g(S_X(t))dt.$$

In this case, $\mathbb{P}$ is the *Choquet integral* with respect to the *distorted probability* $\mu = g \circ \mathbb{P}$. If this happens, $\mathbb{P}$ is said to be a *sublinear Wang's premium principle*. It should be note that a sublinear Wang's premium principle is a *distortion risk measure* (i.e., it is a *coherent risk measure* in the sense of Artzner et al. [1]), which in addition is expressed by means of a Choquet integral (see e.g. Balbàs Garrido and Mayoral [2]).

3 Choquet integral representation of premium functionals

In the following theorem we characterize the representability of a premium functional $\mathbb{P}$ on $L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$ as a Wang's premium principle with respect to a concave probability distortion. It is well known that if $\mathbb{P}$ is a Wang's premium functional on $L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$ with respect to a concave probability distortion g, then $\mathbb{P}$ is subadditive and in particular sublinear (see Denneberg [5]).

We use the following nice result proved by Wirch and Hardy [12].

Lemma 3.1 ([12], Theorem 2.2) *Let $\mathbb{P}$ be a Wang's premium functional on $L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$ with respect to a probability distortion g. If $\mathbb{P}$ is subadditive, then g is concave.*

Theorem 3.2 *Let $\mathbb{P}$ be a sublinear premium functional on $L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$ and assume that $\mathbb{P}$ satisfies the axiom of no unjustified loading. Then the following conditions are equivalent:*

(i) *There exists a concave probability distortion g such that $\mathbb{P}(X) = \int X dg \circ \mathbb{P}$ for all $X \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$;*

(ii) *The following conditions are verified:*

(a) $\mathbb{P}$ *is monotone with respect to stop loss order;*

(b) *for all $X \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$, $c \in \mathbb{R}^+$ and $F \in \mathcal{F}$ such that $\{X > 0\} \subseteq F$:*

$$\mathbb{P}(X + c\mathbb{1}_F) \geq \mathbb{P}(X) + c\mathbb{P}(\mathbb{1}_F).$$

Proof. (i) $\Rightarrow$ (ii). Assume that condition (i) holds. From Dhaene et al. [7, Theorem 8], we have that $\mathbb{P}$ is monotone with respect to stop-loss order. Since $\mathbb{P}$ is comonotone additive, we have that it is widely translation invariant and therefore it is clear that the the above condition (b) holds.
(ii) $\Rightarrow$ (i). Now assume that condition (ii) is verified. Then we have that sublinearity of $\mathbb{P}$ and the above condition (b) immediately imply that $\mathbb{P}$ is widely translation invariant. Since $\mathbb{P}$ is positively homogeneous, widely translation invariant and monotone, it is comonotone additive (see Parker [8, Theorem 7]). Further, from Parker [8, Theorem 9], since $\mathbb{P}(\mathbb{1}_\Omega) = 1$ we have that $\mathbb{P}$ is the Choquet integral with respect to the normalized set function v on $\mathcal{F}$ defined by $v(F) = \mathbb{P}(\mathbb{1}_F)$ for all $F \in \mathcal{F}$. Since in addition $\mathbb{P}$ is monotone with respect to first order stochastic dominance, from Song and Yan [10, Proposition 2.1 and Remark 2.2] there exists a distortion function g such that $\mathbb{P}$ is the Choquet integral of X with respect to the distorted probability $\mu = g \circ \mathcal{P}$. Since $\mathbb{P}$ is subadditive, we have that g is concave from Lemma 3.1. □

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