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On Weak Continuity of Preference Relations with Nontransitive Indifference

Gianni Bosi

Dipartimento di Matematica Applicata
"*Bruno de Finetti*", Università di Trieste
Piazzale Europa 1
34127 Trieste, Italy
GIANNI.BOSI@econ.units.it

Magalì E. Zuanon

Dipartimento di Metodi Quantitativi
Università degli Studi di Brescia
Contrada Santa Chiara 50, 25122 Brescia, Italy

Abstract

We characterize *weak continuity* of an interval order $\precsim$ on a topological space (X, τ) by using the concept of a *scale* in a topological space.

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1 Introduction

An interval order $\precsim$ on a set X is in some sense the simplest kind of binary relation with nontransitive indifference since, under certain conditions, it can be represented by a pair (u, v) of real-valued functions on X (this means that, for all $x, y \in X$, $x \precsim y$ if and only if $u(x) \leq v(y)$). If in addition X is endowed with a topology τ, then one may look for a pair (u, v) of continuous real-valued functions representing an interval order $\precsim$ on (X, τ) (see e.g. Bosi, Candeal and Induráin [2] and Bosi, Candeal, Campión and Induráin [3]).

With a view to possible general conditions guaranteeing the existence of such a continuous representation, Bosi [1] introduced the concept of a *weakly* *continuous* interval order. In this paper, we characterize weak continuity of an interval order by using the concept of a *scale* in a topological space.

2 Notation and preliminaries

We first recall that an *interval order* $\precsim$ on an arbitrary nonempty set X is a binary relation on X which is *reflexive* and in addition verifies the following condition for all $x, y, z, w \in X$:

$$(x \precsim z) \textit{ and } (y \precsim w) \Rightarrow (x \precsim w) \textit{ or } (y \precsim z).$$

The *irreflexive part* of an interval order $\precsim$ will be denoted by $\prec$ (i.e., for all $x, y \in X$, $x \prec y$ if and only if $(x \precsim y)$ and not$(y \precsim x)$).

Fishburn [6] showed that if $\precsim$ is an interval order on a set X, then each of the following two binary relations $\precsim^*$ and $\precsim^{**}$ on X is a *total preorder* (i.e., a *total* and *transitive* binary relation):

$$x \precsim^* y \Leftrightarrow (z \precsim x \Rightarrow z \precsim y) \text{ for all } z \in X,$$

$$x \precsim^{**} y \Leftrightarrow (y \precsim z \Rightarrow x \precsim z) \text{ for all } z \in X.$$

The irreflexive parts of $\precsim^*$ and $\precsim^{**}$ will be denoted by $\prec^*$ and $\prec^{**}$.

If $\precsim$ is an interval order on a set X, then denote by $L_{\prec}(x)$ $(U_{\prec}(x))$ the *strict lower* (*upper*) *section* of any element $x \in X$ (i.e., for every $x \in X$, $L_{\prec}(x) = \{y \in X : y \prec x\}$ and $U_{\prec}(x) = \{y \in X : x \prec y\}$).

A pair (u, v) of real-valued functions on X is said to *represent* an interval order $\precsim$ on X if, for all $x, y \in X$,

$$x \precsim y \Leftrightarrow u(x) \leq v(y).$$

We say that a pair (u, v) of real-valued functions on X *almost represents* an interval order $\precsim$ on X if, for all $x, y \in X$,

$$(x \precsim y \Rightarrow u(x) \leq v(y)) \text{ and } (x \prec y \Rightarrow v(x) \leq u(y)).$$

The following proposition holds which illustrates the importance of the concept of a pair of continuous real-valued functions almost representing an interval order in connection with the problem concerning the existence of a representation by means of a pair of continuous real-valued functions.

Proposition 2.1 *An interval order $\precsim$ on a topological space (X, τ) is representable by means of a pair (u, v) of continuous real-valued functions with* *values in* $[0,1]$ *if and only if there exists a countable family* $\{(u_n, v_v)\}_{n\in\mathbb{N}\setminus\{0\}}$ *of pairs of continuous real-valued functions on* (X,τ) *with values in* $[0,1]$ *almost representing* $\precsim$ *such that for every* $x, y \in X$ *with* $x \prec y$ *there exists* $n \in \mathbb{N}\setminus\{0\}$ *with* $v_n(x) < u_n(y)$.

Proof. The "only if" part is clear. Hence, assume that there exists a countable family $\{(u_n, v_n)\}_{n\in\mathbb{N}\setminus\{0\}}$ of pairs of continuous real-valued functions on (X,τ) with values in $[0,1]$ almost representing $\precsim$ such that for every $x, y \in X$ with $x \prec y$ there exists $n \in \mathbb{N}\setminus\{0\}$ with $v_n(x) < u_n(y)$. Define functions u and v on X as follows:

$$u(x) = \sum_{n=1}^{\infty} 2^{-n} u_n(x), \;\; v(x) = \sum_{n=1}^{\infty} 2^{-n} v_n(x) \qquad (x \in X)$$

in order to immediately verify that (u, v) is a continuous representation of the interval order $\precsim$ on the topological space (X,τ). □

An interval order $\precsim$ on a topological space (X,τ) is said to be *continuous* if $L_{\prec}(x)$ and $U_{\prec}(x))$ are both open subsets of X for every $x \in X$. Further, we say that it is *strongly continuous* if it is continuous and in addition the associated total preorders $\precsim^*$ and $\precsim^{**}$ are both continuous.

We now recall the definition of a *weakly continuous interval order* presented by Bosi [1].

Definition 2.2 (weakly continuous interval order) We say that an interval order $\precsim$ on a topological space (X,τ) is *weakly continuous* if for every $x, y \in X$ such that $x \prec y$ there exists a pair (u_{xy}, v_{xy}) of continuous real-valued functions on (X,τ) satisfying the following conditions:
(i) (u_{xy}, v_{xy}) almost represents $\precsim$;
(ii) $v_{xy}(x) < u_{xy}(y)$.

The concept of weak continuity described in Definition 2.2 is reminiscent of the concept of *weak continuity* of a preorder on a topological space (see e.g. Bosi and Herden [5]). Every interval order that is representable by means of a pair of continuous functions (u, v) and at same time is such that the associated total preorders $\precsim^*$ and $\precsim^{**}$ are not continuous provides an example of a weakly continuous interval order which is continuous but not strongly continuous. For example, this is the case of the interval order $\precsim$ on $X = [3,5] \cup [9,25]$ defined by $x \precsim y \Leftrightarrow x \leq y^2$ (see Bosi, Candeal and Induráin [2, Example 3.2]) when X is endowed with the induced Euclidean topology on the real line.

3 Weak continuity of interval orders

In the sequel, we shall refer to the well known notion of a *scale* in a topological space (see e.g. Gillman and Jerison [7]).

Definition 3.1 If (X, τ) is a topological space and $\mathbb{S}$ is a dense subset of $[0, 1]$ such that $1 \in \mathbb{S}$, then a family $\{G_r\}_{r \in \mathbb{S}}$ of open subsets of X is said to be a *scale* in (X, τ) if the following conditions hold:

(i) $G_1 = X$;

(ii) $\overline{G_{r_1}} \subseteq G_{r_2}$ for every $r_1, r_2 \in \mathbb{S}$ such that $r_1 < r_2$.

We are now ready to characterize the weak continuity of an interval order on a topological space.

Proposition 3.2 *Let $\precsim$ be an interval order on a topological space (X, τ). Then the following conditions are equivalent:*

(i) *$\precsim$ is weakly continuous;*

(ii) *For every pair $(x, y) \in X \times X$ such that $x \prec y$ there exist two scales $\{G_r^{*(xy)}\}_{r \in \mathbb{S}}$ and $\{G_r^{**(xy)}\}_{r \in \mathbb{S}}$ in (X, τ) such that the family $\{(G_r^{*(xy)}, G_r^{**(xy)})\}_{r \in \mathbb{S}}$ satisfies the following conditions:*

  (a) *$z \precsim w$ and $w \in G_r^{*(xy)}$ imply $z \in G_r^{**(xy)}$ for every $z, w \in X$ and $r \in \mathbb{S}$;*

  (b) *$z \prec w$ and $w \in G_r^{**(xy)}$ imply $z \in G_r^{*(xy)}$ for every $z, w \in X$ and $r \in \mathbb{S}$;*

  (c) *$x \in G_r^{*(xy)}$ and $y \notin G_r^{**(xy)}$ for every $r \in \mathbb{S} \setminus \{1\}$.*

Proof. Consider a pair $(x, y) \in X \times X$ such that $x \prec y$.
(i) $\Rightarrow$ (ii). Since $\precsim$ is weakly continuous, there exists a pair (u_{xy}, v_{xy}) of continuous real-valued functions on (X, τ) such that (u_{xy}, v_{xy}) almost represents $\precsim$ and in addition $v_{xy}(x) < u_{xy}(y)$. Without loss of generality, we can assume that both u_{xy} and v_{xy} take values in $[0, 1]$ and that $v_{xy}(x) = 0$, $u_{xy}(y) = 1$. Define $\mathbb{S} = \mathbb{Q} \cap]0, 1]$, $G_r^{*(xy)} = v_{xy}^{-1}([0, r[)$, $G_r^{**(xy)} = u_{xy}^{-1}([0, r[)$ for every $r \in \mathbb{S}$, and $G_1^{*(xy)} = G_1^{**(xy)} = X$ in order to immediately verify that $\{G_r^{*(xy)}\}_{r \in \mathbb{S}}$ and $\{G_r^{**(xy)}\}_{r \in \mathbb{S}}$ are two scales in (X, τ) such that the family $\{(G_r^{*(xy)}, G_r^{**(xy)})\}_{r \in \mathbb{S}}$ satisfies the above conditions (a), (b) and (c).
(ii) $\Rightarrow$ (i). From the assumptions, there exist two scales $\{G_r^{*(xy)}\}_{r \in \mathbb{S}}$ and $\{G_r^{**(xy)}\}_{r \in \mathbb{S}}$ such that the family $\{(G_r^{*(xy)}, G_r^{**(xy)})\}_{r \in \mathbb{S}}$ satisfies the above conditions (a), (b) and (c). Define two functions $u_{xy}, v_{xy} : X \to [0, 1]$ as follows:

$$u_{xy}(z) = inf\{r \in \mathbb{Q} \cap]0, 1] : z \in G_r^{**(xy)}\} \quad (x \in X),$$

$$v_{xy}(z) = inf\{r \in \mathbb{Q} \cap]0,1] : z \in G_r^{*(xy)}\} \quad (x \in X).$$

We have that u_{xy} and v_{xy} are both continuous functions on (X, τ) with values in $[0, 1]$ (see e.g. the proof of the lemma on pages 43-44 in Gillman and Jerison [7]). We claim that the pair (u_{xy}, v_{xy}) almost represents the interval order $\precsim$ and satisfies the condition $v_{xy}(x) < u_{xy}(y)$.
From condition (c), we have that $v_{xy}(x) = 0$ and $u_{xy}(y) = 1$. It remains to show that the pair (u_{xy}, v_{xy}) almost represents the interval order $\precsim$. First consider any two elements $z, w \in X$ such that $z \prec w$. Then, by condition (b), we have that $v_{xy}(z) \leq u_{xy}(w)$. Finally, observe that if $z, w \in X$ are any two elements such that $z \precsim w$, then we have that $u_{xy}(z) \leq v_{xy}(w)$ by condition (a). This consideration completes the proof. □

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