On Weak Continuity of Preference Relations with Nontransitive Indifference

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Abstract

We characterize weak continuity of an interval order \( \succeq \) on a topological space \((X, \tau)\) by using the concept of a scale in a topological space.

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1 Introduction

An interval order \( \succeq \) on a set \( X \) is in some sense the simplest kind of binary relation with nontransitive indifference since, under certain conditions, it can be represented by a pair \((u, v)\) of real-valued functions on \( X \) (this means that, for all \( x, y \in X \), \( x \succeq y \) if and only if \( u(x) \leq v(y) \)). If in addition \( X \) is endowed with a topology \( \tau \), then one may look for a pair \((u, v)\) of continuous real-valued functions representing an interval order \( \succeq \) on \((X, \tau)\) (see e.g. Bosi, Candeal and Induráin [2] and Bosi, Candeal, Campión and Induráin [3]).

With a view to possible general conditions guaranteeing the existence of such a continuous representation, Bosi [1] introduced the concept of a weakly
continuous interval order. In this paper, we characterize weak continuity of an interval order by using the concept of a scale in a topological space.

2 Notation and preliminaries

We first recall that an interval order $\preceq$ on an arbitrary nonempty set $X$ is a binary relation on $X$ which is reflexive and in addition verifies the following condition for all $x, y, z, w \in X$:

$$(x \preceq z) \text{ and } (y \preceq w) \Rightarrow (x \preceq w) \text{ or } (y \preceq z).$$

The irreflexive part of an interval order $\preceq$ will be denoted by $\prec$ (i.e., for all $x, y \in X$, $x \prec y$ if and only if $(x \not\preceq y)$ and not($y \preceq x$)).

Fishburn [6] showed that if $\preceq$ is an interval order on a set $X$, then each of the following two binary relations $\preceq^*$ and $\preceq^{**}$ on $X$ is a total preorder (i.e., a total and transitive binary relation):

$$x \preceq^* y \iff (z \preceq x \Rightarrow z \preceq y) \text{ for all } z \in X,$$

$$x \preceq^{**} y \iff (y \preceq z \Rightarrow x \preceq z) \text{ for all } z \in X.$$

The irreflexive parts of $\preceq^*$ and $\preceq^{**}$ will be denoted by $\prec^*$ and $\prec^{**}$.

If $\preceq$ is an interval order on a set $X$, then denote by $L_<(x)$ ($U_<(x)$) the strict lower (upper) section of any element $x \in X$ (i.e., for every $x \in X$, $L_<(x) = \{y \in X : y < x\}$ and $U_<(x) = \{y \in X : x < y\}$).

A pair $(u, v)$ of real-valued functions on $X$ is said to represent an interval order $\preceq$ on $X$ if, for all $x, y \in X$,

$$x \preceq y \iff u(x) \leq v(y).$$

We say that a pair $(u, v)$ of real-valued functions on $X$ almost represents an interval order $\preceq$ on $X$ if, for all $x, y \in X$,

$$(x \preceq y \Rightarrow u(x) \leq v(y)) \text{ and } (x \prec y \Rightarrow v(x) \leq u(y)).$$

The following proposition holds which illustrates the importance of the concept of a pair of continuous real-valued functions almost representing an interval order in connection with the problem concerning the existence of a representation by means of a pair of continuous real-valued functions.

**Proposition 2.1** An interval order $\preceq$ on a topological space $(X, \tau)$ is representable by means of a pair $(u, v)$ of continuous real-valued functions with
values in \([0, 1]\) if and only if there exists a countable family \(\{(u_n, v_n)\}_{n \in \mathbb{N} \setminus \{0\}}\) of pairs of continuous real-valued functions on \((X, \tau)\) with values in \([0, 1]\) almost representing \(\preceq\) such that for every \(x, y \in X\) with \(x \prec y\) there exists \(n \in \mathbb{N} \setminus \{0\}\) with \(v_n(x) < u_n(y)\).

**Proof.** The “only if” part is clear. Hence, assume that there exists a countable family \(\{(u_n, v_n)\}_{n \in \mathbb{N} \setminus \{0\}}\) of pairs of continuous real-valued functions on \((X, \tau)\) with values in \([0, 1]\) almost representing \(\preceq\) such that for every \(x, y \in X\) with \(x \prec y\) there exists \(n \in \mathbb{N} \setminus \{0\}\) with \(v_n(x) < u_n(y)\). Define functions \(u\) and \(v\) on \(X\) as follows:

\[
u(x) = \sum_{n=1}^{\infty} 2^{-n} u_n(x), \quad v(x) = \sum_{n=1}^{\infty} 2^{-n} v_n(x) \quad (x \in X)
\]

in order to immediately verify that \((u, v)\) is a continuous representation of the interval order \(\preceq\) on the topological space \((X, \tau)\). \(\square\)

An interval order \(\preceq\) on a topological space \((X, \tau)\) is said to be *continuous* if \(L_\prec(x)\) and \(U_\prec(x)\) are both open subsets of \(X\) for every \(x \in X\). Further, we say that it is *strongly continuous* if it is continuous and in addition the associated total preorders \(\preceq^*\) and \(\preceq^{**}\) are both continuous.

We now recall the definition of a *weakly continuous interval order* presented by Bosi [1].

**Definition 2.2 (weakly continuous interval order)** We say that an interval order \(\preceq\) on a topological space \((X, \tau)\) is *weakly continuous* if for every \(x, y \in X\) such that \(x \prec y\) there exists a pair \((u_{xy}, v_{xy})\) of continuous real-valued functions on \((X, \tau)\) satisfying the following conditions:

(i) \((u_{xy}, v_{xy})\) almost represents \(\preceq\);

(ii) \(v_{xy}(x) < u_{xy}(y)\).

The concept of weak continuity described in Definition 2.2 is reminiscent of the concept of *weak continuity* of a preorder on a topological space (see e.g. Bosi and Herden [5]). Every interval order that is representable by means of a pair of continuous functions \((u, v)\) and at same time is such that the associated total preorders \(\preceq^*\) and \(\preceq^{**}\) are not continuous provides an example of a weakly continuous interval order which is continuous but not strongly continuous. For example, this is the case of the interval order \(\preceq\) on \(X = [3, 5] \cup [9, 25]\) defined by \(x \preceq y \iff x \leq y^2\) (see Bosi, Candeal and Induráin [2, Example 3.2]) when \(X\) is endowed with the induced Euclidean topology on the real line.

### 3 Weak continuity of interval orders

In the sequel, we shall refer to the well known notion of a *scale* in a topological space (see e.g. Gillman and Jerison [7]).
**Definition 3.1** If \((X, \tau)\) is a topological space and \(S\) is a dense subset of \([0, 1]\) such that \(1 \in S\), then a family \(\{G_r\}_{r \in S}\) of open subsets of \(X\) is said to be a *scale* in \((X, \tau)\) if the following conditions hold:

(i) \(G_1 = X\);

(ii) \(\overline{G_{r_1}} \subseteq G_{r_2}\) for every \(r_1, r_2 \in S\) such that \(r_1 < r_2\).

We are now ready to characterize the weak continuity of an interval order on a topological space.

**Proposition 3.2** Let \(\preceq\) be an interval order on a topological space \((X, \tau)\). Then the following conditions are equivalent:

(i) \(\preceq\) is weakly continuous;

(ii) For every pair \((x, y)\) in \(X \times X\) such that \(x \prec y\) there exist two scales \(\{G_{r_{xy}}^*(x)\}_{r \in S}\) and \(\{G_{r_{xy}}^{**}(x)\}_{r \in S}\) in \((X, \tau)\) such that the family \(\{(G_{r_{xy}}^*(x), G_{r_{xy}}^{**}(x))\}_{r \in S}\) satisfies the following conditions:

(a) \(z \preceq w\) and \(w \in G_{r_{xy}}^*(x)\) imply \(z \in G_{r_{xy}}^{**}(x)\) for every \(z, w \in X\) and \(r \in S\);

(b) \(z \prec w\) and \(w \in G_{r_{xy}}^{**}(x)\) imply \(z \in G_{r_{xy}}^*(x)\) for every \(z, w \in X\) and \(r \in S\);

(c) \(x \in G_{r_{xy}}^*(x)\) and \(y \notin G_{r_{xy}}^{**}(x)\) for every \(r \in S \setminus \{1\}\).

**Proof.** Consider a pair \((x, y)\) in \(X \times X\) such that \(x \prec y\).

(i) \(\Rightarrow\) (ii). Since \(\preceq\) is weakly continuous, there exists a pair \((u_{xy}, v_{xy})\) of continuous real-valued functions on \((X, \tau)\) such that \((u_{xy}, v_{xy})\) almost represents \(\preceq\) and in addition \(v_{xy}(x) < u_{xy}(y)\). Without loss of generality, we can assume that both \(u_{xy}\) and \(v_{xy}\) take values in \([0, 1]\) and that \(v_{xy}(x) = 0, u_{xy}(y) = 1\).

Define \(S = \mathbb{Q} \cap [0, 1]\), \(G_{r_{xy}}^*(x) = v_{xy}^{-1}([0, r])\), \(G_{r_{xy}}^{**}(x) = u_{xy}^{-1}([0, r])\) for every \(r \in S\), and \(G_{1_{xy}}^*(x) = G_{1_{xy}}^{**}(x) = X\) in order to immediately verify that \(\{G_{r_{xy}}^*(x)\}_{r \in S}\) and \(\{G_{r_{xy}}^{**}(x)\}_{r \in S}\) are two scales in \((X, \tau)\) such that the family \(\{(G_{r_{xy}}^*(x), G_{r_{xy}}^{**}(x))\}_{r \in S}\) satisfies the above conditions (a), (b) and (c).

(ii) \(\Rightarrow\) (i). From the assumptions, there exist two scales \(\{G_{r_{xy}}^*(x)\}_{r \in S}\) and \(\{G_{r_{xy}}^{**}(x)\}_{r \in S}\) such that the family \(\{(G_{r_{xy}}^*(x), G_{r_{xy}}^{**}(x))\}_{r \in S}\) satisfies the above conditions (a), (b) and (c). Define two functions \(u_{xy}, v_{xy} : X \rightarrow [0, 1]\) as follows:

\[
  u_{xy}(z) = \inf\{r \in \mathbb{Q} \cap [0, 1] : z \in G_{r_{xy}}^{**}(x)\} \quad (x \in X),
\]
\[ v_{xy}(z) = \inf \{ r \in \mathbb{Q} \cap [0,1] : z \in G^*_{r}(xy) \} \quad (x \in X). \]

We have that \( u_{xy} \) and \( v_{xy} \) are both continuous functions on \((X, \tau)\) with values in \([0,1]\) (see e.g. the proof of the lemma on pages 43-44 in Gillman and Jerison [7]). We claim that the pair \((u_{xy}, v_{xy})\) almost represents the interval order \( \preceq \) and satisfies the condition \( v_{xy}(x) < u_{xy}(y) \).

From condition (c), we have that \( v_{xy}(x) = 0 \) and \( u_{xy}(y) = 1 \). It remains to show that the pair \((u_{xy}, v_{xy})\) almost represents the interval order \( \preceq \). First consider any two elements \( z, w \in X \) such that \( z \prec w \). Then, by condition (b), we have that \( v_{xy}(z) \leq u_{xy}(w) \). Finally, observe that if \( z, w \in X \) are any two elements such that \( z \preceq w \), then we have that \( u_{xy}(z) \leq v_{xy}(w) \) by condition (a). This consideration completes the proof. \[ \square \]

**References**


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