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Topological A*-Algebras

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Abstract

In this paper we introduce the concept of Topological A^*-algebras and prove every maximal ideal of a Topological A^*-algebra A is closed and every T_2 - Topological A^*-algebra is a Haussdorff- space.

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1. Preliminaries

1.1 Definition: An algebra $(A,\wedge,*,(-)^{\sim},(-)_{\pi},1)$ is an A* - algebra if it satisfies **:**
For a, b, c$\in$A
(i) $a_{\pi} \vee (a_{\pi})^{\sim} = 1$, $(a_{\pi})_{\pi} = a_{\pi}$, where $a\vee b = (a^{\sim}\wedge b^{\sim})^{\sim}$.
(ii) $a_{\pi}\vee b_{\pi} = b_{\pi}\vee a_{\pi}$
(iii) $(a_{\pi}\vee b_{\pi})\vee c_{\pi} = a_{\pi}\vee (b_{\pi}\vee c_{\pi})$
(iv) $(a_{\pi}\wedge b_{\pi})\vee (a_{\pi}\wedge (b_{\pi})^{\sim}) = a_{\pi}$
(v) $(a\wedge b)_{\pi} = a_{\pi}\wedge b_{\pi}$, $(a\wedge b)^{\#} = a^{\#}\vee b^{\#}$, where $a^{\#} = (a_{\pi}\vee a^{\sim}{}_{\pi})^{\sim}$
(vi) $a^{\sim}{}_{\pi} = (a_{\pi}\vee a^{\#})^{\sim}$, $a^{\sim\#} = a^{\#}$
(vii) $(a*b)_{\pi} = a_{\pi}$, $(a*b)^{\#} = (a_{\pi})^{\sim}\wedge(b^{\sim}{}_{\pi})^{\sim}$
(viii) $a = b$ if and only if $a_{\pi} = b_{\pi}$, $a^{\#} = b^{\#}$.
We write 0 for $1^{\sim}$, 2 for $0*1$.

1.2 Example: **3** ={0,1,2} with the operations defined below is an A* -algebra.

$\wedge$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

*	0	1	2
0	0	2	2
1	1	1	1
2	0	2	2

x	0	1	2
$x^{\sim}$	1	0	2
x_{π}	0	1	0

1.3 Definition: A 3–ring is a commutative ring (R, +, .,1) with $x^3 = x$, $3x = 0$ for all x in R .

1.4 Theorem : If If (R, +, ., 1) is a 3-ring then $(R,\wedge, *,(-)^{\sim}, (-)_{\pi}, 1)$ is an A*-algebra, where
(i) $a^{\sim} = 1-a$
(ii) $a\wedge b = 2(1+a)(1+b)[1+(1-a)(1-b)]-1$
(iii) $a_{\pi} = 2a-a^2$
(iv) $a*b = (2a-a^2) + 2(1-a)^2b^2$

1.5Theorem : Let (R,+, .,1) be a 3-ring and $(R, \wedge, *, (-)\sim, (-)\pi, 1)$ be the associated A*-algebra then

(i) $(a+b)_\pi = (a\sim\wedge b)_\pi \vee (a\wedge b^{\sim})_\pi \vee (a^{\#} \wedge b^{\#})$

$(a+b)^{\#} = (a\wedge b)_\pi \vee (a^{\#} \wedge b^{\sim}{}_\pi) \vee (a^{\sim}{}_\pi \wedge b^{\#})$

(ii) $(ab)_\pi = (a\wedge b)_\pi \vee (a^{\#} \wedge b^{\#})$

$(ab)^{\#} = (a_\pi \wedge b^{\#}) \vee (a^{\#} \wedge b_\pi)$

1.6Theorem Let $(A, \wedge, *, (-)^{\sim}, (-)_\pi, 1)$ be an A*-algebra then (A, +, . ,1) is a 3-ring where +, . are defined as follows:

For $a, b \in A$,

$$a+b = (a+b)_\pi * (a+b)^{\#}$$

$$ab = (ab)_\pi * (ab)^{\#}, \text{ Where}$$

$$(a+b)_\pi = (a^{\sim}\wedge b)_\pi \vee (a\wedge b^{\sim})_\pi \vee (a^{\#}\wedge b^{\#})$$

$$(a+b)^{\#} = (a\wedge b)_\pi \vee (a^{\#}\wedge b^{\sim}{}_\pi) \vee (a^{\sim}{}_\pi\wedge b^{\#})$$

$$(ab)_\pi = (a\wedge b)_\pi \vee (a^{\#}\wedge b^{\#})$$

$$(ab)^{\#} = (a_\pi\wedge b^{\#}) \vee (a^{\#} \wedge b_\pi).$$

1.7 Definition: Suppose (G,.) is a group. G is called a Topological group, if there is a Topology $\mathfrak{I}$ on

G such that $. : G\times G\to G$ and $(-)^{-1}: G\to G$ are continuous.

1.8 Note: (i) For $a, b\in G$ and every nbd W of ab, $\exists$ nbds U and V of a, b respectively such that $UV\subseteq W$

where $UV = \{ab/a\in U, b\in V\}$.

(ii) For every nbd W of a^{-1}, $\exists$ a nbd U of a $\ni U^{-1} \subseteq W$.

1.9 Definition: A Topological Ring is a ring R which is also a Topological – Space such that both the addition and multiplication are continuous as maps $R\times R\to R$, where $R\times R$ carries the product topology.

1.10 Definition: A Topological field is a field F in which a Topological ring and inversion is continuous, when restricted to F-{0}.

1.11 Definition: A Topological A^*- algebra A is: an A^*-algebra $(A,\wedge,\vee,*,(-)^{\sim},(-)_{\pi},0,1)$, a Topological Space $(A,\mathfrak{I})$ such that $\wedge,\vee,*$, $(-)_{\pi}$, $(-)^{\sim}$ are continuous with respect to the Topology $\mathfrak{I}$.

1.12 Example: The A^*-algebra **3** = {0, 1, 2} with discrete topology is a Topological A^*-algebra.

1.13 Note: Here after A stands for a Topological A^*- algebra.

1.14 Notations: Suppose X, Y are subsets of A. Then we define

$$X_{\pi} = \{a_{\pi} / a\in X\},$$
$$X^{\sim} = \{a^{\sim} / a\in X\},$$
$$X*Y = \{a*b / a\in X, b\in Y\},$$
$$X\vee Y = \{a\vee b / a\in X, b\in Y\},$$
$$X\wedge Y = \{a\wedge b / a\in X, b\in Y\}.$$

2. Main Results

2.1 Theorem: The following hold in A:

a) If a, b are two elements of A, then for every nbd W of $a\wedge b$, there exists U, V of a, b respectively such that $U\wedge V\subseteq W$.
b) If a,b are two elements of A, then for every nbd W of $a\vee b$, there exists U,V of a,b respectively such that $U\vee V\subseteq W$.
c) If a, b are two elements of A, then for every nbd W of $a*b$, there exist U, V of a, b respectively such that $U*V\subseteq W$.
d) If $a\in A$, then for every nbd W of a_{π} ,$\exists$ a nbd U of such that $U_{\pi} \subseteq W$.
e) If $a\in A$, then for every nbd W of $a^{\sim}$, $\exists$ a nbd U of such that $U^{\sim} \subseteq W$.

Proof:

a) Suppose a, $b\in A$ and W is a nbd of $a\wedge b$.

$\because$ $\wedge: A\times A\rightarrow A$ is continuous, and $a,b\in A$, $\exists$ nbds U and V of a,b respectively such that

$$U\times V \subseteq\wedge^{-1} (W)$$
$$\Rightarrow \wedge (U\times V)\subseteq W$$
$$\Rightarrow \wedge (U\times V)\subseteq W$$

$\Rightarrow U \wedge V \subseteq W$.

b) Suppose $a \in A$ and W is a nbd of $a \vee b$.

$\because \vee : A \times A \rightarrow A$ is continuous , $\exists$ nbds U and V of a,b respectively such that

$U \times V \subseteq \vee^{-1}(W)$

$\Rightarrow \vee(U \times V) \subseteq W$

$\Rightarrow U \vee V \subseteq W$.

c) Suppose $a,b \in A$ and W is a nbd of $a * b$.

$\because * : A \times A \rightarrow$ is continuous, so $\exists$ nbds U,Vof a,b respectively such that $U \times V \subseteq *^{-1}(W)$.

$\Rightarrow *(U \times V) \subseteq W$.

$\Rightarrow U * V \subseteq W$.

d) Suppose $a \in A$ and W is a nbd of a_π .

$\because \pi : A \rightarrow A$ is continuous , $\exists$ a nbd U of a such that $U \subseteq \pi^{-1}(W)$

$\Rightarrow \pi(U) \subseteq W$

$\Rightarrow U_\pi \subseteq W$.

e) Suppose $a \in A$ and W is a nbd of $a^\sim$

$\because \sim : A \rightarrow A$ is continuous, $\exists$ a nbd U of a $\ni U \subseteq \sim^{-1}(W)$

$\Rightarrow \sim(U) \subseteq W$

$\Rightarrow U^\sim \subseteq W$.

2.2 Theorem: Suppose A is a Topological A^*- algebra and $a \in A$. Then the mappings

$f: A \rightarrow A$ by $f(x) = a \vee x$
$g: A \rightarrow A$ by $g(x) = a \wedge x$
$h: A \rightarrow A$ by $h(x) = a * x$
$k: A \rightarrow A$ by $k(x) = x^\sim$
$l : A \rightarrow A$ by $l(x) = x_\pi$ are continuous.

Proof: Since $k = \sim$, $l = \pi$, so k, l are continuous.

$\because \wedge: A \times A \rightarrow A$ is continuous

$\Rightarrow \wedge/\{a\} \times A$ is continuous & $g = \wedge/\{a\}\times A$
$\therefore$ g is continuous .
lly f, h are continuous.

2.3 Note: In A , for $a,b\in A$, from 1.5 Theorem, 1.6 Theorem;
$a = a_\pi * a^\#$, then $a^\# * a_\pi$ is denoted by (-a)
$\therefore -a = a^\# * a_\pi$.

$\therefore (-a)_\pi = a^\#$, $(-a)^\# = a_\pi$, $(-a)^\sim{}_\pi = a^\sim{}_\pi$.
$[(a^\sim{}_\pi \wedge b_\pi)\vee(a_\pi \wedge b^\sim{}_\pi) \vee (a^\#\wedge b^\#)] *[(a_\pi \wedge b_\pi)\vee(a^\#\wedge b^\sim{}_\pi)\vee(a^\sim{}_\pi \wedge b^\#)]$
is denoted by a+b and
$[(a^\sim{}_\pi \wedge b^\#)\vee(a_\pi \wedge b^\sim{}_\pi)\vee(a^\#\wedge b_\pi)]*[(a_\pi \wedge b^\#)\vee(a^\#\wedge b^\sim{}_\pi)\vee(a^\sim{}_\pi \wedge b_\pi)]$ is denoted by a − b.
Clearly a − a = 0, a+b =b+a.
a+ a + a = 0, a + a = -a.
$[(a\wedge b)_\pi \vee (a^\#\wedge b^\#)] * [(a_\pi \wedge b^\#)\vee (a^\#\wedge b_\pi)]$ is denoted by ab.
Clearly ab = ba, 1a = a, 0a = 0.

2.4 Note: In A, $x\to a + x$, $x\to ax$ are homeomorphisms.

2.5 Theorem: If F is a closed set, U is an open set, P is any set, a is an element of A then a F, a+F are closed sets, PU, P+U are open sets.

Proof : Since $x\to ax$ is a homeomorphism, so aF is a closed set.
Since $x\to a + x$ is a homeomorphism, a+F is a closed set.
lly aU, a+U are open sets in A.
$PU = \bigcup_{a\in P} aU$ is open set.
$P+U = \bigcup_{a\in P}(a+U)$ is open set.

2.6 Theorem: Every Topological A^*- algebra is a homogeneous algebra i.e., for every p, q there is a continuous mapping $f: A\to A$ such that f (p) = q.
Proof: Define $f: A\to A$ by f(x) = (q-p) +x.
Then f is continuous and f (p) = q.

2.7 Note: $(A,\wedge,\vee,*,(-)^\sim,(-)_\pi, 0,1)$ is an A^* -algebra .Then (A, +, . ,0,1) is a 3-ring where +, . are as defined in 1.6 Theorem, +, .,– and $\wedge,\vee ,*,(-)^\sim, (-)_\pi$ are equivalent:
For $a,b\in A$,
$a^\sim =1-a$
$a\wedge b = 2(1+a) (1+b)[1+(1-a)(1-b)]-1$
$a_\pi = 2a - a^2$
$a * b =(2a - a^2)+2 (1-a)^2 b^2$.

2.8 Theorem: Suppose K, S are subsets of A, then

a) KS, K+S, K∨S, K∧S, K∗S are compact sets whenever K, S are compact sets.
b) $K^{\sim}$,K_{π} are compact sets whenever K is compact set

c) KS, K+S, K∨S, K∧S, K∗S are connected sets whenever K, S are connected sets.
d) $K^{\sim}$, K_{π} are connected whenever K is connected.

Proof :

a) Since continuous image of a compact set is compact ,
. : A×A →A is continuous, K, S are compact sets i.e., K×S is compact in A×A, so . (K×S) = KS is compact in A.
lly K+S, K∨S, K∧S, K∗S are compact sets.

b) Clear.

c) Since continuous image of a connected set is connected ,
. : A×A →A is continuous, K×S is connected in A×A,
so, . (K×S) = KS is connected set.
lly K+S, K∨S, K∧S, K∗S are connected sets.

d) Clear.

2.9 Theorem: The union of all connected sets containing 0 is a sub A^*-algebra.

Proof: Suppose $\{K_i \,/\, i\in I\}$ is the class of all connected sets such that
$A' = \bigcup_{i \in I} K_i$ contains 0.

$\because 0\in A' \Rightarrow 0\in K_i$ for some $i\in I$

$\because K_i$ is connected, so $K_i^{\sim}$ is connected so $K_i^{\sim}$ is in the class.

$\therefore 1\in A'$.

lly $2\in A'$.

Let $a\in A' \Rightarrow a\in K_i$ for some $i\in I$

$\Rightarrow a^{\sim}\in K_i^{\sim}$

$\Rightarrow a^{\sim}\in A'$.

Suppose $a, b\in A' \Rightarrow a\in K_i, b\in K_j$ for some $i, j \in I$.

$a\wedge b\in K_i \wedge K_j$, $a \vee b \in K_i \vee K_j$, $a*b\in K_i * K_j$,

$a_{\pi}\in K_{i_\pi}$, $a^{\sim} \in K_i^{\sim}$.

$\Rightarrow a\wedge b$, $a \vee b$, $a*b$, a_π, $a^\sim \in A'$, $\because K_i \wedge K_j$,
$K_i \vee K_j$, $K_i * K_j$, K_{i_π} , $K_i^\sim$ are connected.

$\therefore$ A' is a sub A^*- algebra of A.

2.10 Definition: A nonempty subset I of an A^*-algebra A is said to be an A^*-ideal of A if

i) $a,b\in I \Rightarrow a \vee b, a*b\in I$.
ii) $a\in I\Rightarrow a_\pi , a^\#\in I$
iii) $a\in I , b\in A \Rightarrow a_\pi b_\pi , a^\# b^\#\in I$
(Here $xy = x\wedge y$ for all $x,y\in B(A)$)

2.11 Note: (i) In 2.10 (iii), if b =0, then $0\in I$.
(ii) Suppose I is an ideal of A^*-algebra A. For any $a\in A$, We define
$I_a =\{b\in A / a_\pi b_\pi^\sim,\ a_\pi^\sim b_\pi,\ a^\sim{}_\pi b^\sim{}_\pi{}^\sim,\ a^\sim{}_\pi{}^\sim\ b^\sim{}_\pi \in I\}$. Then I_a is called a coset of A with respect to I generated by a and
$I_0 =\{ b\in A / b_\pi ,b^\sim{}_\pi{}^\sim\in I\}$
$I_1 =\{ b\in A / b_\pi^\sim ,b^\sim{}_\pi\in I\}$
$I_2 = \{b\in A / b_\pi ,b^\sim{}_\pi\in I\}$ and $A/I = \{I_a / a\in A\}$.

2.12 Theorem: Suppose A is topological A^*-algebra and I is closed ideal of A.
Then $A/I = \{I_a / a\in A\}$ is a topological A^*-algebra.

Proof : Define $\wedge,\vee,*,(-)^\sim,(-)_\pi$, 0,1,2 in A/I as follows:

$I_a\wedge I_b = I_{a\wedge b}$
$I_a\vee I_b = I_{a\vee b}$
$I_a*I_b = I_{a*b}$
$(I_a)_\pi = I_{a_\pi}$
$(I_a)^\sim = I_{a^\sim}$

$0 = I_0$, $1 = I_1$, $2 = I_2$. Then $(A/I, \wedge,\vee,*,(-)^\sim,(-)_\pi, I_0, I_1,I_2)$ is an A^*-algebra .

Define $f: A\to A/I$ by $f(a) = I_a$ where $a\in I$.
Suppose $a = b \Rightarrow I_a = I_b$.
$\therefore$ f is well defined and clearly f is surjective.
Clearly $f: A\to A/I$ is an A^*- homomorphism.
$\therefore$ $f: A\to A/I$ is an A^*- epimorphism.
Suppose $\Im_I = \{f(U) /U\in\Im\}$.
Since I is closed, $\Im_I$ is a topology on A /I for which $\wedge,\vee,*,(-)^\sim,(-)_\pi$ in A /I are continuous.
$\therefore$ A /I is a Topological A^*- algebra.

2.13 Theorem: Suppose I is an ideal in the Topological A^*-algebra A, then $\overline{I}$ is also an Ideal in A.

Proof: Suppose I is an ideal in the Topological A^*-algebra A.

$\overline{I} = \{a \in A$ / Every nbd of a intersects I $\}$

Claim: $\overline{I}$ is an ideal.

Let $a,b \in \overline{I} \Rightarrow$ Every nbd of a and every nbd of b intersect I.

Let W be a nbd of $a \vee b$.

Then $\exists$ nbds U,V of a, b respectively $\ni U \vee V \subseteq W$.

$\because$ U, V intersect I, so $U \cap V$ intersect I, so W intersect I.

$\therefore a \vee b \in \overline{I}$.

lly $a * b \in \overline{I}$.

Suppose $a \in \overline{I}$, $b \in A$.

Every nbd of a intersects I.

Consider a nbd W of $a_\pi b_\pi$.

$\Rightarrow \exists$ nbd U of ab $\ni U_\pi \subseteq W$

$\Rightarrow \exists$ nbds V, G of a,b $\ni V \wedge G \subseteq U$

$\Rightarrow (V \wedge G)_\pi \subseteq U_\pi$

i.e., $V_\pi \wedge G_\pi \subseteq U_\pi \subseteq W$.

$\because$ V intersects I so V_π intersects I, so $V_\pi \cap G_\pi$.

$\therefore$ W intersects I, $\because V_\pi \wedge G_\pi \subseteq G$.

$\therefore a_\pi b_\pi \in \overline{I}$.

lly $a^\# b^\# \in \overline{I}$.

Clearly $a \in \overline{I} \Rightarrow a_\pi, a^\# \in \overline{I}$.

$\therefore \overline{I}$ is an ideal.

2.14 Theorem: Every maximal ideal M of a Topological A^*-algebra A is closed.

Proof : Clearly $M \subseteq \overline{M}$.

But $\overline{M}$ is an ideal of A.

$\therefore M = \overline{M}$, $\because$ M is maximal.

$\therefore$ M is closed.

2.15 Note: N_a is nbd of a iff $N_a - a$ is a nbd of 0 ($N_a - a = N_a - \{a\}$).

2.16 Theorem: If a Topological A^*-algebra A is T_2 space then it is a Hausdorff Space..

Proof: Let $a, b \in A$ and $a \neq b$.

$\because$ A is T_2-Space, $\exists$ N_a, N_b nbds of a,b respectively $\ni$ $a \notin N_b$, $b \notin N_a$.

Suppose $N_a \cap N_b \neq \phi$.

Let $V = N_a \cap N_b$.

Let $C \in V$ and $C \neq 0$.

Then $U = V - C$ is a nbd of 0.

Let $U_a = U+a$, $U_b = U+b$ then U_a, U_b are nbds of a,b respectively and $U_a \cap U_b = \phi$.

$\therefore$ A is a Hausdorff Space.

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