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Inverse Image and Image of Upper and Lower (α, β)-Fuzzy Set

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Abstract

Using the notion of "belongingness $(\in)$" and "quasi-coincidence (q)" of fuzzy points with fuzzy sets. We introduce the concept of inverse image and image of upper and lower of an (α, β)-fuzzy set, where α and β will denote any one of $\in, q, \in \vee q$ or $\in \wedge q$ with $\alpha \neq \in \wedge q$, and some interesting properties are investigated.

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1 Introduction

The concept of a fuzzy set in a non-empty set was introduced by Zadeh [12] in 1965. Since then, many papers on fuzzy sets appeared showing the importance of the concept and its applications (see, for example, [1, 8, 10]).

The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [9], played a vital role to generate some different types of fuzzy subgroups. Bhakat and Das [3, 4] gave the concepts of (α, β)-fuzzy subgroups by using the notion of "belongingness $(\in)$" and "quasi-coincidence (q)" between a fuzzy point and a fuzzy subgroup, where α, β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$, and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroup. In [5] $(\in, \in \vee q)$- fuzzy subrings and ideals defined. In [7] Jun and Song initiated the study of (α, β)-fuzzy interior ideals of a semigroup. In [2] Bhakat defined $(\in \vee q)$-level subsets of a fuzzy set. In [10] Shabir, Jun et al. studied characterizations of regular semigroups by (α, β)-fuzzy ideals. In [11] Yuan, Li et al. redefined (α, β)-intuitionistic fuzzy subgroups. This paper continues this line of research.

2 Preliminaries

Let X be a non-empty set. A mapping $\mu : X \longrightarrow [0;1]$ is called a *fuzzy set* in X. The *complement* of μ, denoted by μ^c, is the fuzzy set in X given by $\mu^c(x) = 1 - \mu(x)$ for all $x \in X$. Denote by $FS(X)$ the set of all fuzzy sets in X.

For any $t \in [0,1]$ and fuzzy set μ of X, the set

$$U(\mu, t) = \{x \in X | \mu(x) \geq t\} \ \ \text{(respectively, } L(\mu, t) = \{x \in X | \mu(x) \leq t\}),$$

is called an *upper* (respectively, *lower*) *t-level cut* of μ.

Definition 2.1. [8] Let f be a mapping from a set X into a set Y. Let μ be a fuzzy set in X and λ be a fuzzy set in Y. Then the *inverse image* $f^{-1}(\lambda)$ of λ is a fuzzy set in X defined by

$$f^{-1}(\lambda)(x) = \lambda(f(x)) \text{ for all } x \in X.$$

The *image* $f(\mu)$ of μ is the fuzzy set in Y defined by

$$f(\mu)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

for all $y \in Y$.
We have always $f(f^{-1}(\lambda)) \leq \lambda$ and $\mu \leq f^{-1}(f(\mu))$.

Definition 2.2. A fuzzy subset μ of a universe X is a function from X into the unit closed interval $[0,1]$, i.e. $\mu : X \longrightarrow [0,1]$ (see [12]). A fuzzy subset μ in a universe X of the form

$$\mu(y) = \begin{cases} t \in (0,1] & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t. For a fuzzy point x_t and a fuzzy set μ in a set X, Pu and Liu [9] gave meaning to the symbol $x_t \alpha \mu$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$. A fuzzy point x_t is said to belong to (resp. quasi-coincident with) a fuzzy set μ written $x_t \in \mu$ (resp. $x_t q \mu$) if $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$), and in this case, $x_t \in \vee q \mu$ (resp. $x_t \in \wedge q \mu$) means that $x_t \in \mu$ or $x_t q \mu$ (resp. $x_t \in \mu$ and $x_t q \mu$).

In what follows, unless otherwise specified, α and β will denote any one of $\in, q, \in \vee q$ or $\in \wedge q$ with $\alpha \neq \in \wedge q$, which was introduced by Bhakat and Das [4]. To say that $x_t \overline{\alpha} \mu$ means that $x_t \alpha \mu$ does not hold.

3 Inverse Image and Image of Upper and Lower (α, β)-Fuzzy Set

Definition 3.1. [1] Let $t \in (0,1]$ and μ is a fuzzy set in X. We defined
$U(\alpha\mu, t) = \{x \in X |\ x_t\alpha\mu\}$, $L(\in \mu, t) = \{x \in X |\ \mu(x) \leq t\}$,
$L(q\mu, t) = \{x \in X |\ \mu(x) + t \leq 1\})$,
$L(\in \vee q\mu, t) = \{x \in X |\ \mu(x) + t \leq 1$ or $\mu(x) \leq t\}$,
where $\alpha \in \{\in, q, \in \vee q\}$. Then,
(UL_1) the set $U(\in \mu, t)$ and $L(\in \mu, t)$ is called an *upper* and *lower t-level cut* of $\in \mu$, respectively,
(UL_2) the set $U(q\mu, t)$ and $L(q\mu, t)$ is called an *upper* and *lower t-level cut* of $q\mu$, respectively,
(UL_3) the set $U(\in \vee q\mu, t)$ and $L(\in \vee q\mu, t)$ is called an *upper* and *lower t-level cut* of $\in \vee q\mu$, respectively.

It is clear that $U(\in \mu, t) = U(\mu, t)$ and $L(\in \mu, t) = L(\mu, t)$.

Theorem 3.2. [1] Let $\mu \in FS(X)$. Then for all $x \in X$ and $t \in (0,1]$, we have

(1) $U(\in \vee q\mu, t) = U(\in \mu, t) \bigcup U(q\mu, t)$,

(2) $L(\in \vee q\mu, t) = L(\in \mu, t) \bigcup L(q\mu, t)$.

Lemma 3.3. Let $\mu \in FS(X)$. Then for all $x \in X$ and $t \in (0,1]$, we have

(1) $x_t q\mu \Longleftrightarrow x_t \overline{\in} \mu^c$;

(2) $x_t \in \vee q\mu \Longleftrightarrow x_t \overline{\in \wedge q} \mu^c$.

Proof.

(1) Let $x \in X$ and $t \in (0,1]$. Then, we have

$$\begin{aligned} x_t q\mu &\Longleftrightarrow \mu(x) + t > 1 \\ &\Longleftrightarrow 1 - \mu(x) < t \\ &\Longleftrightarrow \mu^c(x) < t \\ &\Longleftrightarrow x_t \overline{\in} \mu^c. \end{aligned}$$

(2) Let $x \in X$ and $t \in (0,1]$. Then, we have

$$\begin{aligned} x_t \in \vee q\mu &\Longleftrightarrow x_t \in \mu \ \text{ or } \ x_t q\mu \\ &\Longleftrightarrow \mu(x) \geq t \ \text{ or } \ \mu(x) + t > 1 \\ &\Longleftrightarrow 1 - \mu^c(x) \geq t \ \text{ or } \ 1 - \mu^c(x) + t > 1 \\ &\Longleftrightarrow x_t \overline{q} \mu^c \ \text{ or } \ x_t \overline{\in} \mu^c \\ &\Longleftrightarrow x_t \overline{\in \wedge q} \mu^c. \end{aligned}$$

□

Definition 3.4. A fuzzy set μ in a set X is said to have sup property if for every non-empty subset S of X, there exists $x' \in S$ such that

$$\mu(x') = \sup_{x \in S}\{\mu(x)\}$$

Theorem 3.5. Let $\mu,\ \lambda \in IFS(X)$ and mapping f from X into Y be a surjection. Let μ and λ have sup property, then for all $t \in (0, 1]$ we have

(1) $U(\alpha f(\mu), t) = f(U(\alpha\mu, t))$,

(2) $L(\alpha f(\lambda), t) \subseteq f(L(\alpha\lambda, t))$,

where $\alpha \in \{\in, q, \in \vee q\}$.

Proof. (1) We only prove the case of $\alpha = q$. The others are analogous.

$$\begin{aligned} y \in U(qf(\mu), t) &\Longleftrightarrow y_t q f(\mu) \\ &\Longleftrightarrow f(\mu)(y) + t > 1 \\ &\Longleftrightarrow \sup_{x \in f^{-1}(y)}\{\mu(x)\} + t > 1 \\ &\Longleftrightarrow \exists x' \in f^{-1}(y),\ \mu(x') + t > 1 \\ &\Longleftrightarrow f(x') = y,\ x'_t q\mu \\ &\Longleftrightarrow f(x') = y,\ x' \in U(q\mu, t) \\ &\Longleftrightarrow y \in f(U(q\mu, t)). \end{aligned}$$

(2) We only prove the case of $\alpha = \in \vee q$.
Let $y \in L(\in \vee q f(\lambda), t)$. Then, we have $f(\lambda)(y) \leq t$ or $f(\lambda)(y) + t \leq 1$. This shows that

$$\sup_{x \in f^{-1}(y)}\{\lambda(x)\} \leq t \text{ or } \sup_{x \in f^{-1}(y)}\{\lambda(x)\} + t \leq 1,$$

and so $\lambda(x) \leq t$ or $\lambda(x) + t \leq 1$ for all $x \in f^{-1}(y)$. This shows that

$$x \in L(\in \vee q\lambda, t) \text{ for all } x \in f^{-1}(y).$$

Therefore $y \in f(L(\in \vee q\lambda, t))$.
The other the cases can be proven analogously. □

Theorem 3.6. Let $\lambda \in FS(X)$ and mapping f from X into Y be a surjection. Let λ and λ^c has sup property, then

(1) $U(\overline{q} f(\lambda), t) = L(qf(\lambda), t) \subseteq U(\in f(\lambda^c), t)$,

(2) $U(\overline{\in} f(\lambda), t) \subseteq L(\in f(\lambda), t) \subseteq U(qf(\lambda^c), t) = L(\overline{q} f(\lambda^c), t)$,

(3) $L(\overline{\in} f(\lambda), t) \subseteq U(\in f(\lambda), t)$,

for all $t \in (0, 1]$.

Proof. (1) Let $y \in L(qf(\lambda), t)$. Then $f(\lambda)(y) + t \leq 1$ and so $\sup_{x \in f^{-1}(y)}\{\lambda(x)\} + t \leq 1$, thus $\lambda(x) + t \leq 1$ for all $x \in f^{-1}(y)$. This shows that $\lambda^c(x) \geq t$ for all $x \in f^{-1}(y)$. Then $\sup_{x \in f^{-1}(y)}\{\lambda^c(x)\} \geq t$. Since λ^c has sup property, then $\exists x' \in f^{-1}(y),\ \lambda^c(x') \geq t$, and so $f(x') = y,\ x'_t \in \lambda^c$. Therefore $y_t \in f(\lambda^c)$, thus $y \in U(\in f(\lambda^c), t)$. Hence, we have $L(qf(\lambda), t) \subseteq U(\in f(\lambda^c), t)$.
Also, $y \in L(qf(\lambda), t)$ if and only if $f(\lambda)(y) + t \leq 1$ if and only if $y \in U(\overline{q} f(\lambda), t)$. Thus $U(\overline{q} f(\lambda), t) = L(qf(\lambda), t)$.

The other the cases can be proven analogously..

□

Theorem 3.7. Let $\mu,\ \lambda \in FS(X)$ and mapping f from X into Y be a map. Then for all $t \in (0, 1]$ we have

(1) $U(\alpha f^{-1}(\mu), t) = f^{-1}(U(\alpha\mu, t))$,

(2) $L(\beta f^{-1}(\lambda), t) = f^{-1}(L(\beta\lambda, t))$,

where $\alpha \in \{\in, q\}$ and $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$.

Proof. (1) Let $\alpha = q$. We have

$$\begin{aligned} x \in U(qf^{-1}(\mu), t) &\iff x_t q f^{-1}(\mu) \\ &\iff f^{-1}(\mu)(x) + t > 1 \\ &\iff \mu(f(x)) + t > 1 \\ &\iff f(x)_t q \mu \\ &\iff f(x) \in U(q\mu, t) \\ &\iff x \in f^{-1}(U(q\mu, t)). \end{aligned}$$

The other the cases of (1) can be proven analogously.

(2) Let $\beta = \in \vee q$. We have

$$\begin{aligned} x \in L(\in \vee q f^{-1}(\lambda), t) &\iff x \in L(\in f^{-1}(\lambda), t) \text{ or } x \in L(qf^{-1}(\lambda), t) \\ &\iff f^{-1}(\lambda)(x) \leq t \text{ or } f^{-1}(\lambda)(x) + t \leq 1 \\ &\iff \lambda(f(x)) \leq t \text{ or } \lambda(f(x)) + t \leq 1 \\ &\iff f(x) \in L(\in \lambda, t) \text{ or } f(x) \in L(q\lambda, t) \\ &\iff f(x) \in L(\in \vee q\lambda, t) \\ &\iff x \in f^{-1}(L(\in \vee q\lambda, t)). \end{aligned}$$

The other the cases of (2) can be proven analogously. □

Theorem 3.8. Let $\mu,\ \lambda \in FS(X)$ and mapping f from X into Y be a map. Then for all $t \in (0,1]$ we have

$$L(\in \vee q f^{-1}(\lambda), t) = f^{-1}(L(\in \lambda, t)) \bigcup f^{-1}(L(q\lambda, t)).$$

Proof. By the proof of Theorem 3.7, we have $x \in L(\in \vee q f^{-1}(\lambda), t)$ if and only if $x \in f^{-1}(L(\in \lambda, t))$ or $x \in f^{-1}(L(q\lambda, t))$ if and only if $x \in (f^{-1}(L(\in \lambda, t)) \bigcup f^{-1}(L(q\lambda, t)))$. □

Theorem 3.9. Let $\mu,\ \lambda \in FS(X)$ and mapping f from X into Y be a map. Then for all $t \in (0,1]$ we have

$$L(\in \wedge q f^{-1}(\lambda), t) = f^{-1}(L(\in \lambda, t)) \bigcap f^{-1}(L(q\lambda, t)).$$

Proof. The proof is similar to the proof of Theorem 3.8. □

Theorem 3.10. Let $\lambda \in FS(X)$ and mapping f from X into Y be a map. Then

(1) $U(\in f^{-1}(\lambda), t) = L(q f^{-1}(\lambda^c), t)$,

(2) $U(q f^{-1}(\lambda), t) = L(\in f^{-1}(\lambda^c), t)$,

for all $t \in (0,1]$.

Proof. (1) Let $x \in L(q f^{-1}(\lambda^c), t)$. Then, if and only if $f^{-1}(\lambda^c)(x) + t \leq 1$ if and only if $\lambda^c(f(x)) + t \leq 1$ if and only if $\lambda(f(x)) \geq t$ if and only if $f^{-1}(\lambda)(x) \geq t$ if and only if $x \in U(\in f^{-1}(\lambda), t)$. Hence, we have $U(\in f^{-1}(\lambda), t) = L(q f^{-1}(\lambda^c), t)$.

(2) The proof is similar to the proof of (1).

□

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