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A New Technique of Initial Boundary Value Problems Using Adomian Decomposition Method

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Abstract

In this paper, a new technique which is applying to treatment of initial boundary value problems by mixed initial and boundary conditions together to obtain a new initial solution at every iteration using Adomian decomposition method (ADM). The structure of a new successive initial solutions can give a more accurate solution.

Keywords: initial boundary value problems, Adomian decomposition method

1. Introduction

Many researchers discussed the initial and boundary value problems. The Adomian decomposition method discussed for solving higher dimensional initial boundary value problems by Wazwaz A. M. [2000]. Analytic treatment for variable coefficient fourth-order parabolic partial differential equations discussed by Wazwaz A. M. [1995, 2001]. The solution of fractional heat-like and wave-like equations with variable coefficients using the decomposition method was found by Momani S. [2005] and so as by using variational iteration method was found by Yulita Molliq R. et.al [2009]. Solving higher dimensional initial boundary value problems by variational iteration decomposition method by Noor M. A. and Mohyud-Din S.T. [2008]. Exact and numerical solutions for non-linear Burger's equation by variational iteration method was applied by Biazar J. and

Aminikhah H. [2009]. Weighted algorithm based on the homotopy analysis method is applied to inverse heat conduction problems and discussed by Shidfar A. and Molabahrami A. [2010]. The boundary value problems was applied by Niu Z. and Wang C. [2010] to calculate a one step optimal homotopy analysis method for linear and nonlinear differential equations with boundary conditions only, and homotopy perturbation technique for solving two-point boundary value problems–compared it with other methods was discussed by Chun C. and Sakthivel R. [2010]. Fractional differential equations with initial boundary conditions by modified Riemann–Liouville derivative was solved by Wu G. and Lee E. W. [2010].

It is interesting to point out that all these researchers obtained the solutions of initial and boundary value problems by using either initial or boundary conditions only. So we present a reliable framework by applying a new technique for treatment initial and boundary value problems by mixed initial and boundary conditions together to obtain a new initial solution at every iterations using Adomian decomposition method. These technique to construct a new successive initial solutions can give a more accurate solution, some examples are given in this paper to illustrate the effectiveness and convenience of this technique.

2. Adomian decomposition method

Adomian [1994] has presented and developed a so-called decomposition method for solving linear or nonlinear problems such as ordinary differential equations. It consists of splitting the given equation into linear and nonlinear parts, inverting the highest-order derivative operator contained in the linear operator on both sides, identifying the initial and/or boundary conditions and the terms involving the independent variable alone as initial approximation, decomposing the unknown function into a series whose components are to be determined, decomposing the nonlinear function in terms of special polynomials called Adomian's polynomials, and finding the successive terms of the series solution by recurrent relation using Adomian's polynomials. Consider the equation

$$F\big(u(x)\big) = g(x), \tag{2.1}$$

where F represents a general nonlinear ordinary differential operator and g is a given function. The linear
terms in $F\big(u(x)\big)$ are decomposed into $Lu + Ru$, where L is an easily invertible operator, which is taken as the highest order derivative and R is the remainder of the linear operator. Thus, Eq. (2.1) can be written as

$$Lu + Ru + Nu = g, \tag{2.2}$$

where Nu represents the nonlinear terms in $F\big(u(x)\big)$. Applying the inverse operator L^{-1} on both sides yields

$$u = \varphi + L^{-1}(g) - L^{-1}(Ru) - L^{-1}(Nu) , \tag{2.3}$$

where φ is the constant of integration satisfies the condition $L\varphi = 0$. Now assuming that the solution u can be represented as infinite series of the form

$$u = \sum_{n=0}^{\infty} u_n , \tag{2.4}$$

Furthermore, suppose that the nonlinear term Nu can be written as infinite series in terms of the Adomian polynomials A_n of the form

$$Nu = \sum_{n=0}^{\infty} A_n , \tag{2.5}$$

where the Adomian polynomials A_n of Nu are evaluated using the formula

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n} N(\sum_{i=0}^{\infty}(\lambda^i u_i))\bigg|_{\lambda=0} , \quad n = 0,1,2,\ldots \tag{2.6}$$

Then substituting (2.4) and (2.5) in (2.3) gives

$$\sum_{n=0}^{\infty} u_n = \varphi + L^{-1}(g) - L^{-1}(R\sum_{n=0}^{\infty} u_n) - L^{-1}(\sum_{n=0}^{\infty} A_n) \tag{2.7}$$

Each term of series (2.4) is given by the recurrent relation

$$\left.\begin{array}{ll} u_0 = \varphi + +L^{-1}(g), & n = 0 \\ u_{n+1} = L^{-1}(Ru_n) - L^{-1}(A_n). & n \geq 0 \end{array}\right\} \tag{2.8}$$

The convergence of the decomposition series have investigated by several authors [Cherrualt Y. (1989), Cherrualt Y. and Adomian G. (1993)].

3. Adomian decomposition method for solving initial boundary value problems

To convey the basic idea for treatment of initial and boundary conditions by Adomian decomposition method for solving initial boundary value problems, we consider the following one dimensional differen-tial equation

$$F\big(u(x,t)\big) = g(x,t), \quad 0 < x < 1, t > 0, \tag{3.1}$$

the initial conditions associated with (3.1) are of the form

$$u(x,0) = f_0(x), \ \frac{\partial u(x,0)}{\partial t} = f_1(x), \quad 0 \le x \le 1, \tag{3.2}$$

and the boundary conditions are given by

$$u(0,t) = g_0(t), \qquad u(1,t) = g_1(t), \quad t > 0, \tag{3.3}$$

where $f_0(x),\ f_1(x), g_0(t)\ and\ g_1(t)$ are given functions. The initial solution can be written as

$$u_0(x,t) = f_0(x) + t f_1(x). \tag{3.4}$$

We construct a new successive initial solutions u_n^* at every iteration for Eq. (3.1) by applying a new technique

$$u_n^*(x,t) = u_n(x,t) + (1-x)[g_0(t) - u_n(0,t)] + x\,[g_1(t) - u_n(1,t)], \tag{3.5}$$

where $n = 0,1,2,\dots$, clearly that the new successive initial solutions u_n^* of Eq. (3.1) satisfying the initial and boundary conditions together as follows

$$\begin{aligned} &if\ \ t = 0\ \ then\ \ u_n^*(x,0) = u_n(x,0),\\ &if\ \ x = 0\ \ then\ \ u_n^*(0,t) = g_0(t),\\ &if\ \ x = 1\ \ then\ \ u_n^*(1,t) = g_1(t). \end{aligned} \tag{3.6}$$

generally, sometimes a new technique of formula (3.5) can be applying for higher dimensional initial boundary value problems by mixed initial and boundary conditions together to have a new successive initial solutions u_n^* as follows

Consider the k dimensional equation

$$F\big(u(x_1,x_2,\dots,x_k,t)\big) = g(x_1,x_2,\dots,x_k,t), \quad 0 < x_1,x_2,\dots,x_k < 1, t > 0, \tag{3.7}$$

Then we have

$$\begin{aligned} u_n^*(x_1,x_2,\dots,x_k,t) \\ &= u_n(x_1,x_2,\dots,x_k,t) \\ &+ (1-x_1)[g_{01}(x_2x_3,\dots,x_k,t) - u_n(0,x_2,\dots,x_k,t)] \\ &+ x_1[g_{11}(x_2x_3,\dots,x_k,t) - u_n(1,x_2,\dots,x_k,t)] + \cdots \\ &+ (1-x_k)[g_{0k}(x_1x_2,\dots,x_{k-1},t) - u_n(x_1,x_2,\dots,x_{k-1},0,t)] \\ &+ x_k[g_{1k}(x_1,x_2,\dots,x_{k-1},t) - u_n(x_1,x_2,\dots,x_{k-1},1,t)], \end{aligned} \tag{3.8}$$

where $n = 0,1,2, \dots$, and the boundary conditions associated with Eq. (3.7) are of the form

$$u(0,x_2,\dots,x_k,t) = g_{01}(x_2,x_3,\dots,x_k,t), \qquad u(1,x_2,\dots,x_k,t) = g_{11}(x_2,x_3,\dots,x_k,t),$$

$$u(x_1,0,\dots,x_k,t) = g_{02}(x_1,x_3,\dots,x_k,t), \qquad u(x_1,1,\dots,x_k,t) = g_{12}(x_1,x_3,\dots,x_k,t),$$

$$\vdots$$

$$u(x_1,\dots,x_{k-1},0,t) = g_{0k}(x_1,\dots,x_{k-1},t), \qquad u(x_1,\dots,x_{k-1},1,t) = g_{1k}(x_1,\dots,x_{k-1},t), \tag{3.9}$$

and the initial conditions are given by

$$u(x_1,x_2,\dots,x_k,0) = f_0(x), \qquad \frac{\partial u(x_1,x_2,\dots,x_k,0)}{\partial t} = f_1(x). \tag{3.10}$$

Such as treatment of formula (3.8) when $k = 2$ is a very effective as shown in this paper.

4. Applications and results

Example 1: Consider the following one-dimensional heat-like problem [Momani S.(2005)]

$$\frac{\partial u}{\partial t} - \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2} = 0, \;\; 0 < x < 1, t > 0, \tag{4.1}$$

subject to the initial conditions

$$u(x,0)=x^2, \qquad 0<x<1, \tag{4.2}$$

and the boundary conditions

$$u(0,t)=0\,, \quad u(1,t)=\mathrm{e}^{\mathrm{t}}\,, \quad t>0. \tag{4.3}$$

By applying a new approximations u_n^* in Eq. (3.5) we obtain

$$u_n^*(x,t)=u_n(x,t)+(1-x)[0-u_n(0,t)]+x[\mathrm{e}^{\mathrm{t}}-u_n(1,t)], \tag{4.4}$$

where $n=0,1,2,\dots$. The initial approximation is $u_0(x,t)=u(x,0)=x^2$.

Now, we begin with a new initial approximation u_0^* $(when\ n=0)$

$$u_0^*(x,t)=x^2+x\,(\mathrm{e}^{\mathrm{t}}-1). \tag{4.5}$$

According to the Adomian decomposition method [Adomian G., 1994; Wazwaz A.M., 2001], we have an operater form for Eq.(4.1) as

$$Lu=\frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}\,, \qquad 0<x<1, t>0, \tag{4.6}$$

where the differential operator is $L=\frac{\partial}{\partial t}$, so that L^{-1} is integral operator

$$L^{-1}(.)=\int_0^t (.)dt. \tag{4.7}$$

By operating with L^{-1}on both sides of Eq.(4.6) and using a new technique of initial solutions u_n^* we have

$$u_{n+1}(x,t)=\int_0^t\left(\frac{1}{2}x^2\frac{\partial^2\,u_n^*(x,t)}{\partial x^2}\right)dt. \tag{4.8}$$

By Eq. (4.5) we have

$$u_1(x,t)=x^2t\,,$$

And by Eq. (4.4) we have consequently the following approximants

$$u_2(x,t)=\frac{x^2t^2}{2!}\,,$$

$$u_3(x,t) = \frac{x^2t^3}{3!}, \tag{4.9}$$

and so on, the series solution is given by

$$u(x,t) = x^2\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right), \tag{4.10}$$

and in a closed form by

$$u(x,t) = x^2e^t. \tag{4.11}$$

Which is an exact solution.

***Example 2*:** We next consider the one-dimensional wave-like equation[Momani S.(2005)]

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 < x < 1, t > 0, \tag{4.12}$$

subject to the initial conditions

$$u(x,0) = x, \quad \frac{\partial u(x,0)}{\partial t} = x^2, \quad 0 < x < 1, \tag{4.13}$$

and the boundary conditions

$$u(0,t) = 0 \quad , \quad u(1,t) = 1 + \sinh t\,, \quad t > 0. \tag{4.14}$$

By applying a new approximations u_n^* in Eq. (3.5) we obtain

$$u_n^*(x,t) = u_n(x,t) + (1-x)[0 - u_n(0,t)] + x[1 + \sinh t - u_n(1,t)], \tag{4.15}$$

where $n = 0,1,2, \ldots$. The initial approximation is

$$u_0(x,t) = u(x,0) + \frac{\partial u(x,0)}{\partial t}\, t = x + x^2t. \tag{4.16}$$

Now, we begin with a new initial approximation u_0^* $(when\; n = 0)$

$$u_0^*(x,t) = x + x^2t + x\,(\sinh t - t). \tag{4.17}$$

By Adomian decomposition method , we have an operator form for Eq.(4.12) as

$$Lu = \frac{1}{2}x^2 \frac{\partial^2 u}{\partial x^2}\ , \qquad 0 < x < 1, t > 0, \tag{4.18}$$

where the differential operator is $L = \frac{\partial^2}{\partial t^2}$, so that L^{-1} is a two-fold integral operator

$$L^{-1}(.) = \int_0^t \int_0^t (.)dtdt. \tag{4.19}$$

By operating with L^{-1} on both sides of (4.18) and using a new technique of initial solutions u_n^* we have

$$u_{n+1}(x,t) = \int_0^t \int_0^t \left(\frac{1}{2}x^2 \frac{\partial^2\ u_n^*(x,t)}{\partial x^2}\right) dtdt\ , \qquad n \geq 0. \tag{4.20}$$

By Eq. (4.17) we have

$$u_1(x,t) = \frac{x^2 t^3}{3!}\ ,$$

And by Eq. (4.15) we have consequently the following approximants

$$u_2(x,t) = \frac{x^2 t^5}{5!},$$

$$u_3(x,t) = \frac{x^2 t^7}{7!}, \tag{4.21}$$

and so on, the series solution is given by

$$u(x,t) = x + x^2\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \cdots\right), \tag{4.22}$$

and in a closed form by

$$u(x,t) = x + x^2 \sinh t. \tag{4.23}$$

Which is an exact solution.

Example 3: Consider the two-dimensional initial boundary value problem [Noor M.A. and Mohyud-Din S.T.(2008)]:

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2}y^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}x^2 \frac{\partial^2 u}{\partial y^2}, \qquad 0 < x, y < 1, t > 0, \tag{4.24}$$

and the initial conditions

$$u(x,y,0)=x^2+y^2,\quad \frac{\partial u(x,y,0)}{\partial t}=-(x^2+y^2), \tag{4.25}$$

with boundary conditions

$$\begin{aligned} &u(0,y,t)=y^2e^{-t}, \quad u(1,y,t)=(1+y^2)e^{-t},\\ &u(x,0,t)=x^2e^{-t}, \quad u(x,1,t)=(1+x^2)e^{-t}. \end{aligned} \tag{4.26}$$

By applying a new approximations u_n^* in Eq. (3.8) we obtain

$$u_n^*(x,y,t)=u_n(x,y,t)+(1-x)[y^2e^{-t}-u_n(0,y,t)]+x[(1+y^2)e^{-t}-u_n(1,y,t)]+ (1-y)[x^2e^{-t}-u_n(x,0,t)]+y[(1+x^2)e^{-t}-u_n(x,1,t)], \tag{4.27}$$

where $n=0,1,2,\ldots$. The initial approximation is

$$u_0(x,y,t)=x^2+y^2-(x^2+y^2)t. \tag{4.28}$$

Now, we begin with a new initial approximation u_0^* $(when\ n=0)$

$$\begin{aligned} u_0^*(x,y,t)&=x^2+y^2-(x^2+y^2)t+(1-x)[y^2e^{-t}-y^2+y^2t]\\ &\quad+x[(1+y^2)e^{-t}-(1+y^2)+(1+y^2)t]+(1-y)[x^2e^{-t}-x^2+x^2t]\\ &\quad+y[(1+x^2)e^{-t}-(x^2+1)+(x^2+1)t],\\ &=-x-y+xe^{-t}+ye^{-t}+x^2e^{-t}+y^2e^{-t}+(x+y)t. \end{aligned} \tag{4.29}$$

According to the Adomian decomposition method, we have an operator form for Eq.(4.24) as

$$Lu=\frac{1}{2}y^2\frac{\partial^2 u}{\partial x^2}+\frac{1}{2}x^2\frac{\partial^2 u}{\partial y^2}, \qquad 0<x<1, t>0, \tag{4.30}$$

where the differential operator is $L=\frac{\partial^2}{\partial t^2}$, so that L^{-1} is a two-fold integral operator

$$L^{-1}(.)=\int_0^t\int_0^t(.)dtdt. \tag{4.31}$$

By operating with L^{-1} on both sides of (4.30) and using a new technique of initial solutions u_n^* we have the following iteration formula

$$u_{n+1}^*(x,y,t)=\int_0^t\int_0^t\left(\frac{1}{2}y^2\frac{\partial^2 u_n^*(x,y,t)}{\partial x^2}+\frac{1}{2}x^2\frac{\partial^2 u_n^*(x,y,t)}{\partial y^2}\right)dtdt. \tag{4.32}$$

By Eq. (4.29) we obtain a first iteration

$$u_1(x,y,t) = -x^2 - y^2 + (x^2+y^2)t + (x^2+y^2)e^{-t}. \tag{4.33}$$

We can readily check

$$u(x,y,t) = u_0(x,y,t) + u_1(x,y,t) = (x^2+y^2)e^{-t}. \tag{4.34}$$

Which yields an exact solution of Eq. (4.24).

***Example 4*:** Consider the two-dimensional nonlinear inhomogeneous initial boundary value problem [Noor M.A. and Mohyud-Din S.T. (2008)]:

$$\frac{\partial^2 u}{\partial t^2} = \frac{15}{2}\left(x\,u_{xx}^2 + y\,u_{yy}^2\right) + 2x^2 + 2y^2, \qquad 0 < x, y < 1, t > 0, \tag{4.35}$$

and the initial conditions

$$u(x,y,0) = 0, \quad \frac{\partial u(x,y,0)}{\partial t} = 0, \tag{4.36}$$

with boundary conditions

$$\begin{aligned} &u(0,y,t) = y^2t^2 + yt^6, \qquad u(1,y,t) = (1+y^2)t^2 + (1+y)t^6,\\ &u(x,0,t) = x^2t^2 + xt^6, \qquad u(x,1,t) = (1+x^2)t^2 + (1+x)t^6. \end{aligned} \tag{4.37}$$

By applying a new approximations u_n^* in Eq. (3.8) we obtain

$$\begin{aligned} u_n^*(x,y,t) = u_n(x,y,t) &+ (1-x)[y^2t^2 + yt^6 - u_n(0,y,t)]\\ &+ x[(1+y^2)t^2 + (1+y)t^6 - u_n(1,y,t)]\\ &+ (1-y)[x^2t^2 + xt^6 - u_n(x,0,t)]\\ &+ y[(1+x^2)t^2 + (1+x)t^6 - u_n(x,1,t)], \quad n = 0,1,2,\ldots \end{aligned} \tag{4.38}$$

The initial approximation is $u_0(x,y,t) = u(x,y,0) + \frac{\partial u(x,y,0)}{\partial t}\ t = 0.$

Now, we begin with a new initial approximation u_0^* $(when\ n = 0)$

$$\begin{aligned} u_0^*(x,y,t) = (x^2+y^2)t^2 &+ (1-x)yt^6 + x(1+y)t^6 + (1-y)xt^6\\ &+ y(1+x)t^6, \end{aligned} \tag{4.39}$$

By Adomian decomposition method, we have an operator form for Eq.(4.35) as

$$Lu = \frac{15}{2}\left(x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2}\right) + 2x^2 + 2y^2\,, \qquad 0 < x < 1, t > 0, \tag{4.40}$$

where the differential operator is $L = \frac{\partial^2}{\partial t^2}$, so that L^{-1} is a two-fold integral operator

$$L^{-1}(.) = \int_0^t \int_0^t (.)dtdt. \tag{4.41}$$

By operating with L^{-1} on both sides of (4.40) the components $u_n(x, y, t)$ can be determind as

$$u_0(x, y, t) = u(x, y, 0) + \frac{\partial u(x, y, 0)}{\partial t}\; t + L^{-1}(2x^2 + 2y^2)$$

$$= \int_0^t \int_0^t (2x^2 + 2y^2)\; dt\; dt = (x^2 + y^2)t^2, \tag{4.42}$$

$$u_{n+1}(x, y, t) = \int_0^t \int_0^t \left(\frac{15}{2}\left(x\frac{\partial^2 u_n(x, y, t)}{\partial x^2} + y\frac{\partial^2 u_n(x, y, t)}{\partial y^2}\right)\right)dtdt\,, \quad n \geq 0. \tag{4.43}$$

By a new technique of initial solutions u_n^*, we have the following iteration formula

$$u_{n+1}^*(x, y, t) = \int_0^t \int_0^t \left(\frac{15}{2}\left(x\frac{\partial^2 u_n^*(x, y, t)}{\partial x^2} + y\frac{\partial^2 u_n^*(x, y, t)}{\partial y^2}\right)\right)dtdt. \tag{4.44}$$

By Eq. (4.39) we obtain a first iteration

$$u_1(x, y, t) = (x + y)t^6. \tag{4.45}$$

We can readily check

$$u(x, y, t) = u_0(x, y, t) + u_1(x, y, t) = (x^2 + y^2)t^2 + (x + y)t^6. \tag{4.46}$$

Which yields an exact solution of Eq. (4.35).

Example 5: Consider the two-dimensional initial boundary value problem [Wazwaz A.M. (2009)]:

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}y^2\frac{\partial^2 u}{\partial y^2}, \qquad 0 < x, y < 1, t > 0, \tag{4.47}$$

and the initial conditions

$$u(x, y, 0) = x^2 + y^2, \quad \frac{\partial u(x, y, 0)}{\partial t} = y^2 - x^2, \tag{4.48}$$

with boundary conditions

$$u(0, y, t) = y^2 e^t, \qquad u(1, y, t) = e^{-t} + y^2 e^t,$$
$$u(x, 0, t) = x^2 e^{-t}, \qquad u(x, 1, t) = e^t + x^2 e^{-t}. \tag{4.49}$$

By applying a new approximations u_n^* in Eq. (3.8) we obtain

$$u_n^*(x, y, t) = u_n(x, y, t) + (1 - x)[y^2 e^t - u_n(0, y, t)] + x[e^{-t} + y^2 e^t - u_n(1, y, t)] + (1 - y)[x^2 e^{-t} - u_n(x, 0, t)] + y[e^t + x^2 e^{-t} - u_n(x, 1, t)], \tag{4.50}$$

where $n = 0,1,2, \ldots$. The initial approximation is

$$u_0(x, y, t) = x^2 + y^2 + (y^2 - x^2)t. \tag{4.51}$$

Now, we begin with a new initial approximation u_0^* $(when\ n = 0)$

$$u_0^*(x, y, t) = x^2 + y^2 + (y^2 - x^2)t + (1 - x)[y^2 e^t - y^2 - y^2 t] + x[e^{-t} + y^2 e^t - 1 - y^2 - (y^2 - 1)t] + (1 - y)[x^2 e^{-t} - x^2 + x^2 t] + y[e^t + x^2 e^{-t} - x^2 - 1 - (1 - x^2)t]. \tag{4.52}$$

According to the Adomian decomposition method, we have an operator form for Eq.(4.47) as

$$Lu = \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}y^2\frac{\partial^2 u}{\partial y^2}, \qquad 0 < x < 1, t > 0, \tag{4.53}$$

where the differential operator is $L = \frac{\partial^2}{\partial t^2}$, so that L^{-1} is a two-fold integral operator

$$L^{-1}(.) = \int_0^t \int_0^t (.)dtdt. \tag{4.54}$$

By operating with L^{-1} on both sides of (4.53) and using a new technique of initial solutions u_n^* we have the following iteration formula

$$u_{n+1}^*(x,y,t) = \int_0^t \int_0^t \left(\frac{1}{2}x^2 \frac{\partial^2 u_n^*(x,y,t)}{\partial x^2} + \frac{1}{2}y^2 \frac{\partial^2 u_n^*(x,y,t)}{\partial y^2}\right) dtdt. \quad (4.55)$$

By Eq. (4.52) we obtain a first iteration

$$u_1(x,y,t) = -x^2 - y^2 - (y^2 - x^2)t + x^2 e^{-t} + y^2 e^t. \quad (4.56)$$

We can readily check

$$u(x,y,t) = u_0(x,y,t) + u_1(x,y,t) = x^2 e^{-t} + y^2 e^t. \quad (4.57)$$

Which yields an exact solution of Eq. (4.47).

Example 6: Consider the two-dimensional initial boundary value problem [Wazwaz A.M. (2009)]:

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2}x^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}y^2 \frac{\partial^2 u}{\partial y^2}, \qquad 0 < x,y < 1, t > 0, \quad (4.58)$$

and the initial conditions

$$u(x,y,0) = y^2, \quad \frac{\partial u(x,y,0)}{\partial t} = x^2, \quad (4.59)$$

with boundary conditions

$$u(0,y,t) = y^2 \cosh t, \qquad u(1,y,t) = \sinh t + y^2 \cosh t,$$
$$u(x,0,t) = x^2 \sinh t, \qquad u(x,1,t) = x^2 \sinh t + \cosh t. \quad (4.60)$$

By applying a new approximations u_n^* in Eq. (3.8) we obtain

$$u_n^*(x,y,t) =$$
$$u_n(x,y,t) + (1-x)[y^2 \cosh t - u_n(0,y,t)] + x[\sinh t + y^2 \cosh t - u_n(1,y,t)] +$$
$$(1-y)[x^2 \sinh t - u_n(x,0,t)] + y[x^2 \sinh t + \cosh t - u_n(x,1,t)], \quad (4.61)$$

where $n = 0,1,2,\ldots$. The initial approximation is

$$u_0(x,y,t) = y^2 + x^2 t. \quad (4.62)$$

Now, we begin with a new initial approximation u_0^* $(when\ n = 0)$

$$u_0^*(x,y,t) = y^2 + x^2t + (1-x)[y^2\cosh t - y^2] + x[\sinh t + y^2\cosh t - y^2 - t] + (1-y)[x^2\sinh t - x^2t] + y[x^2\sinh t + \cosh t - 1 - x^2t]. \quad (4.63)$$

According to the Adomian decomposition method, we have an operator form for Eq.(4.58) as

$$Lu = \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}y^2\frac{\partial^2 u}{\partial y^2}, \qquad 0 < x < 1, t > 0, \quad (4.64)$$

where the differential operator is $L = \frac{\partial^2}{\partial t^2}$, so that L^{-1} is a two-fold integral operator

$$L^{-1}(.) = \int_0^t\int_0^t (.)dtdt. \quad (4.65)$$

By operating with L^{-1} on both sides of (4.64) and using a new technique of initial solutions u_n^* we have the following iteration formula

$$u_{n+1}^*(x,y,t) = \int_0^t\int_0^t \left(\frac{1}{2}x^2\frac{\partial^2 u_n^*(x,y,t)}{\partial x^2} + \frac{1}{2}y^2\frac{\partial^2 u_n^*(x,y,t)}{\partial y^2}\right)dtdt. \quad (4.66)$$

By Eq. (4.63) we obtain a first iteration

$$u_1(x,y,t) = -y^2 - x^2t + x^2\sinh t + y^2\cosh t. \quad (4.67)$$

We can readily check

$$u(x,y,t) = u_0(x,y,t) + u_1(x,y,t) = x^2\sinh t + y^2\cosh t. \quad (4.68)$$

Which yields an exact solution of Eq. (4.58).

5. Conclusions

In this paper, a very effective to construct a new initial successive solutions u_n^* by mixed initial and boundary conditions together which explained in formula (3.5) and (3.8) when $k = 2$ which is used to find successive approximations u_n of the solution u by Adomian decomposition method to solve initial boundary value problems. These technique to construct of a new successive initial solutions can give a more accurate solution. Some examples (linear and nonlinear, homogeneous and nonhomogeneous problems) are given in this paper to illustrate the effectiveness and convenience of a new technique. It is important and obvious that the exact solutions have found directly from a first iteration for the last four examples but if we use initial conditions only we will have exact solution by calculating infinite successive solutions u_n which closed form by Eq. (2.4).

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