On Parallel Surfaces of Ruled Surfaces with Null Ruling in Minkowski 3-Space

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Abstract

Dillen and Kühnel worked on ruled Weingarten surfaces in Minkowski 3-space [4]. In this study, it is shown that parallel surfaces of ruled surfaces with null ruling are also Weingarten surfaces. Also the relation $H^r = K^r$ between $K^r$ and $H^r$ Gaussian and mean curvatures of parallel surface has been obtained for any parallel surface of ruled surface with null ruling.

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1 Introduction

In theory of surfaces, there are some special surfaces such as ruled surfaces, minimal surfaces and surfaces of constant curvature in which differential geometers are interested. Among these surfaces, parallel surfaces have also been studied in many paper [2, 3, 6, 7, 14].

A surface $M^r$ whose points are at a constant distance along the normal from another surface $M$ is said to be parallel to $M$. So, there are infinite number of parallel surfaces because we choose the constant distance along the
normal arbitrarily. A parallel surface can be regarded as the locus of points which are on the normals to $M$ at a non-zero constant distance $r$ from $M$.

Weingarten property of surfaces has been discussed in many papers [1, 4, 5, 9, 10, 11, 15, 17]. In this paper, we will study the parallel surface of ruled surfaces with null ruling in Minkowski 3-space $E^3_1$ satisfying the following condition

$$\frac{\partial K^r}{\partial u} \frac{\partial H^r}{\partial v} - \frac{\partial K^r}{\partial v} \frac{\partial H^r}{\partial u} = 0 \quad (1)$$

where $K^r$ and $H^r$ are the Gaussian curvature and the mean curvature of the parallel surface of a ruled surface with null ruling, respectively. If the parallel surface of the ruled surface satisfies the equation (1), then the parallel surface is said to be parallel Weingarten surface.

2 Preliminaries

Let $E^3_1$ be a 3-dimensional Lorentzian space with Lorentzian metric

$$ds^2 = dx_1^2 + dx_2^2 - dx_3^2$$

If $\langle X, Y \rangle = 0$, $X$ and $Y$ are called perpendicular in the sense of Lorentzian, where $\langle , \rangle$ is the induced inner product in $E^3_1$. The norm of $X \in E^3_1$ is denoted by $\|X\|$ and defined as

$$\|X\| = \sqrt{|\langle X, X \rangle|}. \quad (2)$$

Let $X \in E^3_1$ be a vector. If $\langle X, X \rangle < 0$, then $X$ is called time-like; if $\langle X, X \rangle > 0$, then $X$ is called space-like and if $\langle X, X \rangle = 0$, $X \neq 0$, then $X$ is called light-like (null) vector. We can observe that a time-like curve corresponds to the path of an observer moving at less than the speed of light while the space-like curves faster and the null curves are equal to the speed of light.

A surface in the Minkowski 3-space is called a time-like surface if the induced metric on the surface is a Lorentzian metric, i.e., the normal on the surface is a space-like vector. A time-like ruled surface in Minkowski 3-space is obtained by a time-like straight line moving a space-like curve or by a space-like straight line moving a time-like curve.

**Definition 2.1** A ruled surface $M$ in Minkowski 3-space is a regular surface that has a parametrization $\varphi : U \to M$ of the form

$$\varphi(u, v) = \alpha(u) + vX(u)$$

where $\alpha$ and $X$ are curves in $E^3_1$ with $\alpha'$ never 0. We call $\varphi$ a ruled patch. The curve $\alpha$ is called the directrix or base curve of the ruled surface, and $X$ is called the director curve [4], [8], [11].
On parallel surfaces of ruled surfaces

The rulings of the ruled surface are the straight lines $v \rightarrow \alpha(u) + vX(u)$. In this study, we will take the ruling $X$ as null, that is, $\langle X(u), X(u) \rangle = 0$. It can be easily seen that ruled surfaces with null ruling are timelike surfaces.

**Theorem 2.2** Using standard parameters, a ruled surface is up to Lorentzian motions, uniquely determined by the following quantities:

\[
Q = \langle \alpha', X \wedge X' \rangle \\
J = \langle X, X'' \wedge X' \rangle \\
F = \langle \alpha', X \rangle
\]

Each of which is a function of $u$. Conversely, every choice of these three quantities uniquely determines a ruled surface [11].

**Theorem 2.3** Any ruled surface with a null ruling is a Weingarten surface satisfying the equation $H^2 = K$ [4].

**Definition 2.4** Let $M$ be a surface in Minkowski 3-space and $D$ be the Levi-Civita connection on Minkowski 3-space. Then,

\[
S : \chi(M) \rightarrow \chi(M), X \rightarrow S(X) = D_X N
\]

is called the shape operator, where $N$ is the unit normal vector on $M$ [7].

**Definition 2.5** Let $M$ be a surface in Minkowski 3-space and let $S$ be shape operator of $M$, for $p \in M$, $K$ and $H$ denote Gaussian and mean curvature of $M$, respectively, and these curvature functions are defined as

\[
K : M \rightarrow \mathbb{R} \\
\quad \quad \quad p \mapsto K(p) = \varepsilon \det S_p \\
H : M \rightarrow \mathbb{R} \\
\quad \quad \quad p \mapsto H(p) = \varepsilon \text{trace}S_p
\]

respectively, where $\varepsilon = <N, N> = \pm 1$ and $N$ is the unit normal vector field on $M$ [13].

**Theorem 2.6** Gaussian and mean curvatures of surface $M$ in Minkowski 3-space are

\[
H = \varepsilon \frac{eg - f^2}{2(EG - F^2)}
\]

and

\[
K = \varepsilon \frac{eg - f^2}{EG - F^2}
\]

respectively, where $\varepsilon = <N, N> = \pm 1$ and $N$ is the unit normal vector field on $M$ [12].
3 Parallel surfaces in Minkowski 3-space

First, we obtain the representation of points on $M^r$ using the representations of points on $M$. Let $\varphi$ be the position vector of a point $P$ on $M$ and $\varphi^r$ be the position vector of a point $f(P)$ on the parallel surface $M^r$. Then $f(P)$ is at a constant distance $r$ from $P$ along the normal to the surface $M$. Therefore the parametrization for $M$ is given by

$$\varphi^r(u,v) = \varphi(u,v) + rN(u,v)$$

where $r$ is a constant scalar and $N$ is the unit normal vector field on $M$ [8].

**Definition 3.1** Let $M$ and $M^r$ be two surfaces in Minkowski 3-space. The function

$$f : M \rightarrow M^r$$

$$p \rightarrow f(p) = p + rN_p$$

is called the parallelization function between $M$ and $M^r$ and furthermore $M^r$ is called parallel surface to $M$ in $E^3_1$ where $r$ is a given positive real number and $N$ is the unit normal vector field on $M$ [6].

**Theorem 3.2** Let $M$ be a surface and $M^r$ be a parallel surface of $M$ in Minkowski 3-space. Let $f : M \rightarrow M^r$ be the parallelization function. Then for $X \in \chi(M)$,

1) $f_*(X) = X + rS(X)$
2) $S_r(f_*(X)) = S(X)$
3) $f$ preserves principal directions of curvature, that is

$$S_r(f_*(X)) = \frac{k}{1+rk}f_*(X)$$

where $S_r$ is the shape operator on $M^r$, and $k$ is a principal curvature of $M$ at $p$ in direction $X$ [7].

**Theorem 3.3** Let $M$ be a timelike surface and $M^r$ be a parallel surface of $M$ in Minkowski 3-space. Then we have

$$K^r = \frac{K}{1+2rH + r^2K}$$
$$H^r = \frac{H + rK}{1+2rH + r^2K}$$

where Gaussian and mean curvatures of $M$ and $M^r$ be denoted by $K$, $H$ and $K^r$, $H^r$ respectively [16].

**Corollary 3.4** Let $M$ be a timelike surface and $M^r$ be a parallel surface of $M$ in Minkowski 3-space. Then we have

$$K = \frac{K^r}{1-2rH + r^2K^r}$$
$$H = \frac{H^r - rK^r}{1-2rH^r + r^2K^r}$$

where Gaussian and mean curvatures of $M$ and $M^r$ be denoted by $K$, $H$ and $K^r$, $H^r$ respectively [16].
4 Weingarten property of surfaces which are parallel to ruled surfaces with null ruling

**Definition 4.1** Let \( M^r \) be a parallel surface to a surface \( M \) in Minkowski 3-space. If there is a nontrivial relation as

\[
\Phi(K^r, H^r) = 0
\]

(12)

between the Gaussian curvature \( K^r \) and the mean curvature \( H^r \) of the parallel surface \( M^r \), the parallel surface \( M^r \) is said to be Weingarten surface.

**Theorem 4.2** Let \( M^r \) be a parallel surface to a surface \( M \) in Minkowski 3-space. If the parallel surface is Weingarten surface, then it satisfies the following condition

\[
\frac{\partial K^r}{\partial u} \frac{\partial H^r}{\partial v} - \frac{\partial K^r}{\partial v} \frac{\partial H^r}{\partial u} = 0
\]

(13)

where \( K^r \) and \( H^r \) are the Gaussian curvature and the mean curvature of the parallel surface of ruled surface with null ruling, respectively.

**Proof.** If Jacobi determinant is taken as a functional relation between \( K^r \) Gauss and \( H^r \) mean curvatures of parallel surface \( M^r \) and is vanished, Weingarten property (13)

\[
\Phi(K^r, H^r) = \det \begin{pmatrix} K^r_u & K^r_v \\ H^r_u & H^r_v \end{pmatrix} = K^r_u H^r_v - K^r_v H^r_u = 0
\]

(14)

is obtained. □

**Theorem 4.3** Let \( \varphi^r \) be a timelike parallel surface of a ruled surface \( \varphi \) in Minkowski 3-space. If \( \varphi \) is a Weingarten surface if and only if \( \varphi^r \) is a Weingarten surface.

**Proof.** (\( \Rightarrow \)): If \( \varphi \) is a timelike ruled Weingarten surface, then it satisfies the following equation

\[
K_u H_v - K_v H_u = 0.
\]

(15)

By using (15), we will try to see the following equation

\[
K^r_u H^r_v - K^r_v H^r_u = 0.
\]

(16)

By substituting (10) into (16), then the equation (16) will become as follows

\[
K^r_u H^r_v - K^r_v H^r_u = \frac{[K_u H_v - K_v H_u]}{(1 + 2rH + r^2k)} \{1 + 2rH + r^2K + 2r^3KH\}.
\]

(17)
By substituting (15) into (17), the equation (17) is obtained as the following expression
\[ K^r_u H^r_v - K^r_v H^r_u = 0. \] (18)

(18) means the parallel surface \( \varphi^r \) is a Weingarten surface.

\((\Leftarrow)\) : If \( \varphi^r \) is timelike Weingarten surface which is parallel to a ruled surface, then it satisfies the following equation
\[ K^r_u H^r_v - K^r_v H^r_u = 0. \] (19)

By using (19), we will try to see the following equation
\[ K^r_u H^r_v - K^r_v H^r_u = 0. \] (20)

So,
\[ K^r_u H^r_v - K^r_v H^r_u = \frac{1}{(1 + 2r H^r - r^2 K^r)^4} \{ K^r_u H^r_v - r K^r_v K^r_u \\ - K^r_v H^r_u + r K^r_u K^r_v - 2r H^r K^r_u H^r_v + 2r H^r K^r_v H^r_u \\ - r^2 K^r K^r_u H^r_v + r^2 K^r K^r_v H^r_u \}. \] (21)

If some terms are eliminated in (21), then the equation (21) will become as follows
\[ K^r_u H^r_v - K^r_v H^r_u = \frac{(K^r_u H^r_v - K^r_v H^r_u)}{(1 - 2r H^r + r^2 K^r)^4} \{ 1 - 2r H^r + r^2 K^r \}. \] (22)

By substituting (19) into (22), the equation (22) is obtained as the following expression
\[ K^r_u H^r_v - K^r_v H^r_u = 0. \] (23)

(23) means the ruled surface \( \varphi \) is a Weingarten surface. \( \blacksquare \)

**Corollary 4.4.** A parallel surface to any ruled surface with null ruling in Minkowski 3-space is a Weingarten surface.

**Proof.** Ruled surface with null ruling in Minkowski 3-space can be parametrized as follows
\[ \varphi(u, v) = \alpha(u) + v X(u) \] (24)
\[ \langle \alpha', \alpha' \rangle = \varepsilon, \quad \langle X, X \rangle = 0, \quad \langle X', X' \rangle = 1. \] (25)

By using (24) and (25), coefficients of the first fundamental form \( I \) are obtained as follows
\[ E = \langle \varphi_u, \varphi_u \rangle = \varepsilon + v^2 \]
\[ F = \langle \varphi_u, \varphi_v \rangle = \langle \alpha', X' \rangle \] (26)
\[ G = \langle \varphi_v, \varphi_v \rangle = 0. \]
Normal vector is \( N = \alpha' \wedge X + vX' \wedge X' \). Coefficients of the second fundamental form \( II \) are obtained as follows

\[
e = \langle \varphi_{uu}, N \rangle = \langle \alpha'', \alpha' \wedge X \rangle + v \langle \alpha'', X' \wedge X \rangle + v \langle X'', \alpha' \wedge X \rangle + v^2 \langle X'', X' \wedge X \rangle
\]

By substituting (3) of Theorem 2.2 into (28) and (28) expression is obtained as follows

\[
f = \langle \varphi_{uv}, N \rangle = \langle \alpha', X \wedge X' \rangle.
\]

By substituting (26), (27), (29) and (30) into (5) and (6) of Theorem 2.6, the followings

\[
K = Q \frac{F}{r} \quad \text{and} \quad H = Q \frac{1}{r}
\]

are obtained. By using (31), it can be seen that ruled surface with null ruling is a Weingarten surface because \( K \) Gauss and \( H \) mean curvatures depend on one parameter \( u \). By substituting (31) into (10) of Theorem 3.3, the followings

\[
K^r = Q \frac{F^2}{r - 2rQ + r^2Q^2} \quad \text{and} \quad H^r = Q \frac{rQ^2}{F - 2rQ + r^2Q^2}
\]

are obtained. By using (32), it can be seen that parallel surface of ruled surface with null ruling is a Weingarten surface because \( K^r \) Gauss and \( H^r \) mean curvatures of parallel surface depend on one parameter \( u \).

**Corollary 4.5** Let \( \varphi^r \) be a timelike parallel surface of a ruled surface \( \varphi \) in a Minkowski 3-space. In this case, there is a relation as follows

\[
H^r^2 = K^r
\]

between the Gaussian curvature \( K^r \) and the mean curvature \( H^r \) of the parallel Weingarten surface.

**Proof.** From Theorem 2.3, we know that there is a relation between \( K \) Gauss and \( H \) mean curvatures of ruled surface with null ruling as follows

\[
H^2 = K.
\]

By substituting (11) of Corollary 3.4 into (34), we have

\[
\left( \frac{H^r - rK^r}{1 - 2rH^r + r^2K^r} \right)^2 = \frac{K^r}{1 - 2rH^r + r^2K^r} \Rightarrow H^r^2 = K^r.
\]
Example 4.6  Let’s give 2-dimensional ivor for instance. This surface is parametrized as follows

\[ \varphi(u,v) = (v \cosh u, v \sinh u, u + v). \]  (35)

This surface is a ruled surface because it can be written down as

\[ \varphi(u,v) = (0, 0, u) + v(\cosh u, \sinh u, 1) \]  (36)

\[ \langle \alpha', \alpha' \rangle = 1, \langle \beta, \beta \rangle = 0. \]  (37)

Under some special parameters, this surface is seen in Figure 1.

![Figure 1](image)

\[ K \text{ Gaussian curvature and } H \text{ mean curvature of timelike surface are calculated as follows} \]

\[ K = 1 \text{ and } H = -1. \]  (38)

By substituting (38) into (10) of Theorem 3.3, Gaussian curvature and mean curvature of parallel surface

\[ K^r = \frac{1}{(1+r)^2} \text{ and } H^r = \frac{1}{1+r} \]  (39)

are obtained, respectively. Parallel surface of timelike 2-dimensional ivor surface is also Weingarten one because partial derivatives of \( K^r \) Gauss and \( H^r \) mean curvatures of parallel surface according to variables \( u \) and \( v \) become 0. Parallel Weingarten surface is parameterized as follows,

\[ \varphi^r(u,v) = (v \cosh u + r \sinh u - vr \cosh u, v \sinh u + r \cosh u - vr \sinh u, u + v - rv). \]  (40)

Under some special parameters, Original surface (red one) and its parallel surface (blue one) are seen in Figure 2.
On parallel surfaces of ruled surfaces

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