Differential Subordinations Associated with Multiplier Transformations

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Abstract. The object of the present paper is to derive some properties of analytic functions in the open unit disc which are defined by using multiplier transformations, applying differential subordinations techniques.

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1. INTRODUCTION

Let $A(p, n)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{j=p+n}^{\infty} a_j z^j \quad (p, n \in \mathbb{N} = \{1, 2, 3\ldots\}),$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. In particular, we set $A(p, 1) = A_p$, $A(1, n) = A(n)$ and $A(1, 1) = A = A_1$, which are well known classes of analytic functions in $U$.

We consider the following multiplier transformation.

Definition 1.1([4]). Let $f(z) \in A(p, n)$. The extended multiplier transformation $I_{\lambda} \left( \delta, \lambda, l \right)$ on $A(p, n)$ is defined by the following infinite series:
\[ I_p(\delta, \lambda, l)f(z) = z^p + \sum_{j=p+1}^{\infty} \frac{p + \lambda(j - p) + l}{p + l} a_j z^j, \]

where \( p, n \in \mathbb{N}, \delta, \lambda, l \in \mathbb{R}, \delta \geq 0, \lambda \geq 0, \text{and } l \geq 0. \)

It follows from (1.1) that \( I_p(0, \lambda, l)f(z) = f(z) \) and
\[ (p + l)I_p(\delta, \lambda, l)f(z) = (p(1 - \lambda) + l)I_p(\delta, \lambda, l)f(z) + \lambda z (I_p(\delta, \lambda, l)f(z))'. \]

**Remark 1.2.** For \( n = 1 \), the operator \( D^\delta = I_i(\delta, 1, 0) \) was introduced and studied by Bhoosnurmath and Swamy [3], which reduces to the Salagean differential operator [10], when \( \delta = m, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \) The operator \( D^\delta_{\lambda,m} = I_i(m, \lambda, 0) \) was introduced and studied by Al-Oboudi [2]. The operator \( I^m_i = I_i(m, 1, l) \) was studied recently by Cho and Srivastava [6] and by Cho and Kim [7]. The operator \( D^\delta_j = I_i(\delta, \lambda, 0) \) was introduced by Acu and Owa [1] and the operator \( I_p(m, 1, l) \) was investigated by Shivaprasad Kumar et al. [11]. The operator \( I^m_i = I_i(m, 1, l) \) was studied by Uralegaddi and Somanatha [12].

Mostafa and Aouf [9] and Catas et al. [5] have obtained many interesting results associated with the multiplier operator \( I_p(\delta, \lambda, l). \)

The main object of this paper is to present some more interesting properties of analytic functions defined by using multiplier transformations \( I_p(\delta, \lambda, l)f(z) \) associated with the class \( A(p, n). \)

In order to prove our main results, we will make use of the following lemma.

**Lemma 1.3** [8]. Let \( \Omega \) be a set in the complex plane \( \mathbb{C}. \) Suppose that the function \( \Psi : \mathbb{C} \times U \to \mathbb{C} \) satisfies the condition \( \Psi(ix_2, y_1; z) \notin \Omega \) for all \( z \in U \) and for all real \( x_2 \) and \( y_1 \) such that
\[ y_1 \leq -\frac{1}{2} n (1 + x_2^2). \]

If \( p(z) = 1 + c_n z^n + \ldots \) is analytic in \( U \) and for \( z \in U, \psi(p(z), z p'(z); z) \subset \Omega, \) then \( \text{Re}(p(z)) > 0 \) in \( U. \)

**2. MAIN RESULTS**

**Theorem 2.1.** Let \( \alpha \) be a complex number satisfying \( \text{Re}(\alpha) > 0 \) and \( \rho < 1. \) Let \( \delta, \lambda, l \in \mathbb{R}, \delta \geq 0, \lambda \geq 0, l \geq 0, f(z), g(z) \in A(p, n) \) and
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\[ \Re \left\{ \frac{I_p(\delta, \lambda, l)g(z)}{I_p(\delta + 1, \lambda, l)g(z)} \right\} > \gamma, 0 \leq \gamma < \Re(\alpha), z \in U. \]

Then

\[ \Re \left\{ \frac{I_p(\delta, \lambda, l)f(z)}{I_p(\delta, \lambda, l)g(z)} \right\} > \frac{2(p + l)\rho + \lambda n\gamma}{2(p + l) + \lambda n\gamma}, z \in U, \]

whenever

\[ \Re \left\{ (1 - \alpha) \frac{I_p(\delta, \lambda, l)f(z)}{I_p(\delta, \lambda, l)g(z)} + \alpha \frac{I_p(\delta + 1, \lambda, l)f(z)}{I_p(\delta + 1, \lambda, l)g(z)} \right\} > \rho, z \in U. \]

Proof. Let \( \tau = (2(p + l)\rho + \lambda n\gamma)/(2(p + l) + \lambda n\gamma) \) and define the function \( p(z) \) by

\[ p(z) = (1 - \tau)^{-1} \left\{ \frac{I_p(\delta, \lambda, l)f(z)}{I_p(\delta, \lambda, l)g(z)} - \tau \right\}. \]

Then, clearly, \( p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \ldots \) and is analytic in \( U \). We set

\[ u(z) = \frac{\alpha I_p(\delta, \lambda, l)g(z)}{I_p(\delta + 1, \lambda, l)g(z)} \]

and observe from (2.1) that \( \Re(u(z)) > \gamma, z \in U \). Making use of the identity (1.2), we find from (2.3) that

\[ (2.4) \left\{ (1 - \alpha) \frac{I_p(\delta, \lambda, l)f(z)}{I_p(\delta, \lambda, l)g(z)} + \alpha \frac{I_p(\delta + 1, \lambda, l)f(z)}{I_p(\delta + 1, \lambda, l)g(z)} \right\} = \tau + (1 - \tau)[p(z) + \frac{\lambda u(z)}{p + l}zp'(z)]. \]

If we define \( \psi(x, y; z) \) by

\[ \psi(x, y; z) = \tau + (1 - \tau) \left( x + \frac{\lambda u(z)}{p + l} y \right), \]

then, we obtain from (2.2) and (2.4) that

\[ \{ \psi(p(z), zp'(z); z) : |z| < 1 \} \subset \Omega = \{ w \in \mathbb{C} : \Re(w) > \rho \}. \]

Now for all \( z \in U \) and for all real \( x_2 \) and \( y_1 \) constrained by the inequality (1.3), we find from (2.5) that

\[ \Re \{ \psi(ix_2, y_1; z) \} = \tau + (1 - \tau) \frac{\lambda y_1}{p + l} \Re(u(z)) \leq \tau - (1 - \tau) \frac{\lambda n\gamma}{2(p + l)} \equiv \rho. \]

Hence \( \psi(ix_2, y_1; z) \notin \Omega \). Thus by Lemma 1.1, \( \Re(p(z)) > 0 \) and hence
Re \( \left( \frac{I_p(\delta, \lambda, l)f(z)}{I_p(\delta, \lambda, l)g(z)} \right) > \tau \) in \( U \). This proves our theorem.

If we set
\[
v(z) = \frac{I_p(\delta+1, \lambda, l)f(z)}{I_p(\delta+1, \lambda, l)g(z)} + \left( \frac{1}{\alpha} - 1 \right) \frac{I_p(\delta, \lambda, l)f(z)}{I_p(\delta, \lambda, l)g(z)},
\]
then for \( \delta \geq 0, \lambda \geq 0, l \geq 0, \alpha > 0 \) and \( \rho = 0 \), Theorem 2.1 reduces to

\[
(2.6) \quad \text{Re}(v(z)) > 0, z \in U \quad \text{implies} \quad \text{Re} \left( \frac{I_p(\delta, \lambda, l)f(z)}{I_p(\delta, \lambda, l)g(z)} \right) > \frac{\lambda n\alpha \gamma}{2(p+l) + \lambda n\alpha \gamma}, z \in U,
\]
whenever

\[
\text{Re} \left( \frac{I_p(\delta, \lambda, l)g(z)}{I_p(\delta+1, \lambda, l)g(z)} \right) > \gamma, 0 \leq \gamma \leq 1, z \in U. \quad \text{Let} \quad \alpha \to \infty. \quad \text{Then} \quad (2.6) \quad \text{is equivalent to}
\]

\[
\text{Re} \left( \frac{I_p(\delta+1, \lambda, l)f(z)}{I_p(\delta+1, \lambda, l)g(z)} - \frac{I_p(\delta, \lambda, l)f(z)}{I_p(\delta, \lambda, l)g(z)} \right) > 0 \quad \text{in} \quad U
\]
implies

\[
\text{Re} \left( \frac{I_p(\delta, \lambda, l)f(z)}{I_p(\delta, \lambda, l)g(z)} \right) > 1 \quad \text{in} \quad U, \quad \text{whenever} \quad \text{Re} \left( \frac{I_p(\delta, \lambda, l)g(z)}{I_p(\delta+1, \lambda, l)g(z)} \right) > \gamma, 0 \leq \gamma \leq 1, z \in U.
\]

In the following theorem we shall extend the above result, the proof of which is similar to that of Theorem 2.1.

**Theorem 2.2.** Let \( \delta, \lambda, l \in \mathbb{R}, \delta \geq 0, \lambda \geq 0, l \geq 0, \rho < 1, f(z), g(z) \in A(p, n) \) and
\[
\text{Re} \left( \frac{I_p(\delta, \lambda, l)g(z)}{I_p(\delta+1, \lambda, l)g(z)} \right) > \gamma, 0 \leq \gamma < 1, z \in U. \quad \text{If}
\]
\[
\text{Re} \left( \frac{I_p(\delta+1, \lambda, l)f(z)}{I_p(\delta+1, \lambda, l)g(z)} - \frac{I_p(\delta, \lambda, l)f(z)}{I_p(\delta, \lambda, l)g(z)} \right) > -\frac{\lambda n\gamma(1-\rho)}{2(p+l)}, z \in U,
\]
then
\[
\text{Re} \left( \frac{I_p(\delta, \lambda, l)f(z)}{I_p(\delta, \lambda, l)g(z)} \right) > \rho, z \in U.
\]
In a manner similar to Theorem 2.1, we can easily prove the following theorems:

**Theorem 2.3.** Let \( \delta, \lambda, l \in R, \delta \geq 0, \lambda \geq 0, l \geq 0, \mu > 0, \rho < 1 \) and \( f(z) \in A(p,n) \). Then for \( \alpha \) a complex number with \( \text{Re}(\alpha) > 0 \), we have
\[
\text{Re}\left( \frac{I_p(\delta, \lambda, l)f(z)}{z^p} \right)^\mu > \frac{2\mu(p + l)\rho + \lambda \text{Re}(\alpha)}{2\mu(p + l) + \lambda n \text{Re}(\alpha)}, z \in U,
\]
whenever
\[
\text{Re}\left\{ (1 - \alpha)\left( \frac{I_p(\delta, \lambda, l)f(z)}{z^p} \right)^\mu + \alpha \left( \frac{I_p(\delta + 1, \lambda, l)f(z)}{I_p(\delta, \lambda, l)f(z)} \right) \right\} > \rho, z \in U.
\]

**Theorem 2.4.** Let \( \delta, \lambda, l \in R, \delta \geq 0, \lambda \geq 0, l \geq 0, \mu > 0, \alpha \) a complex number with \( \text{Re}(\alpha) > 0 \) and \( \frac{\lambda n \text{Re}(\alpha)}{2\mu(p + l) + \lambda n \text{Re}(\alpha)} \leq \rho < 1 \). If \( f(z) \in A(p,n) \) satisfies the condition
\[
\text{Re}\left( \frac{I_p(\delta + 1, \lambda, l)f(z)}{z^p} \right)^\mu + \alpha \left( \frac{I_p(\delta + 1, \lambda, l)f(z)}{I_p(\delta, \lambda, l)f(z)} \right) \right\} > M(p,n,\alpha,\lambda,l,\mu,\rho),
\]
\( (z \in U) \), then
\[
\text{Re}\left( \frac{I_p(\delta, \lambda, l)f(z)}{z^p} \right)^\mu > \rho, z \in U,
\]
where
\[
M(p,n,\alpha,\lambda,l,\mu,\rho) = \rho \left[ (2\mu(p + l) + \lambda n \text{Re}(\alpha))\rho - \lambda n \text{Re}(\alpha) \right] / 2\mu(p + l).
\]

\( \rho = \frac{\lambda n \text{Re}(\alpha)}{2\mu(p + l) + \lambda n \text{Re}(\alpha)} \) and \( \rho = \frac{\lambda n \text{Re}(\alpha)}{2[2\mu(p + l) + \lambda n \text{Re}(\alpha)]} \) in Theorem 2.4

yields the following:

**Corollary 2.5.** Let \( \delta, \lambda, l \in R, \delta \geq 0, \lambda \geq 0, l \geq 0, \mu > 0, \alpha \) a complex number with \( \text{Re}(\alpha) > 0 \) and \( f(z) \in A(p,n) \). Then
\( \text{(a)} \)
\[
\text{Re}\left( (1 - \alpha)\left( \frac{I_p(\delta, \lambda, l)f(z)}{z^p} \right)^\mu + \alpha \left( \frac{I_p(\delta + 1, \lambda, l)f(z)}{I_p(\delta, \lambda, l)f(z)} \right) \right\} > 0, z \in U
\]
implies
\[
\text{Re}\left( \frac{I_p(\delta, \lambda, l)f(z)}{z^p} \right)^\mu > \frac{\lambda n \text{Re}(\alpha)}{2\mu(p + l) + \lambda n \text{Re}(\alpha)}, z \in U,
\]
and

\[
\text{Re}\left( (1-\alpha) \left( I_\gamma \left( \frac{(\delta, \lambda, l) f(z)}{z^p} \right) \right)^{2\mu} + \alpha \left[ I_\gamma \left( \frac{(\delta+1, \lambda, l) f(z)}{I_\gamma (\delta, \lambda, l) f(z)} \right) \right]\right) > M(p, n, \alpha, \lambda, l, \mu), \ z \in U
\]

implies

\[
\text{Re}\left( \left( \frac{I_\gamma (\delta, \lambda, l) f(z)}{z^p} \right)^{\mu} \right) > \frac{\lambda n \text{Re}(\alpha)}{2[(2\mu(p + l) + \lambda n \text{Re}(\alpha))]}, \ z \in U,
\]

where

\[
M(p, n, \alpha, \lambda, l, \mu) = -\frac{\lambda^2 n^2 (\text{Re}(\alpha))^2}{8\mu(p + l)[2\mu(p + l) + \lambda n \text{Re}(\alpha)]}.
\]
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