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Generalized Conditional Convergence and Common Fixed Point Principle for Operators on Metric Spaces

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Abstract

The present paper deals with a generalised common fixed point principle on metric spaces under a generalised conditional convergence of the sequence generated by p self mappings which is more general in nature than that of the orbits of mappings in the fixed point principle given by Som 1984 and Leader 1977 in turn. The results obtained generalise all the previous results of Som 1984 in the sense of the convergence of the sequence to the common fixed point.

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Introduction

Leader [1] has given a fixed point priciple based on uniform equivalence of orbits generated by the self mapping defined on the metric space (X, d), which was further generalised by Som and Mukherjee [2] for two mappings. The main principles derived in Leader [1] and Som and Mukherjee [2] are simple and limited to two mappings and can not be applied to more than two mappings under iteration jointly and generating the sequence as such the derivation becomes quite tedious when we have p-number of operators under consideration.

The condition which played a vital role for the existence of the fixed point in the above papers is given by

$$d(f^n x, f^n y) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

for all x, y in X. In this paper this condition is replaced by a new conditional convergence on the sequence generated by p self mappings under a new structure defined below. The given new structure is more general than the sequence generated by previous authors.
Let (f_l, X, d), $l = 1, 2, \cdots p$ be p operators on a metric space (X, d). Let $x_0 \in X$. Consider the sequence generated by x_0 by the application of the operators $(f_l, X, d), l = 1, 2, \cdots p$ in the following manner.
$x_1 = f_1 x_0, x_2 = f_2 x_1 \cdots \cdots x_p = f_p x_{p-1}$
In general

$$\begin{aligned} x_n &= f_n(x_{n-1}) \qquad \text{if} \quad n \leq p \\ &= f_q(x_{n-1}) \qquad \text{if} \quad n = tp + q \\ &= f_p(x_{n-1}) \qquad \text{if} \quad n = tp \end{aligned} \tag{1}$$

The limit z of this sequence must be fixed under certain weak conditions (for example $f_l \quad l = 1, 2, \cdots p$ have closed graphs or $d(x_0, f_l(x_0)) \quad l = 1, 2, \cdots p$ are lower semi continuous). We call this fixed point z as generalized fixed point if it is the limit of every sequence $\{y_n\}$ with arbitrary initial point $\{y_0\}$ defined in the same manner as x_n.

Results :

Now we give our first result using the sequence $\{x_n\}$ generated through p-operators as mentioned in [1].
Theorem 1 : Let (f_l, X, d), $l = 1, 2, \cdots p$ be p operators on a metric space (X, d). Let $\{x_n\}$ and $\{y_n\}$ be two sequences defined as in (1) with initial points x_0 and y_0 respectively. Let ϵ_i, $i \in N$, be positive real numbers defined by

$$\epsilon_n = \text{Sup}\{d(x_i, y_i) : i \geq n, d(x_0, y_0) \leq c\} \tag{2}$$

If

$$(m/p) \sum_{s=0}^{p-1} \epsilon_{n+s} + \sum_{s=0}^{p-1} \epsilon_{n+s} - (q/p) \sum_{s=0}^{p-1} \epsilon_{n+s} + \epsilon_m \leq c$$

and $d(x, f_l(x)) \leq c$ for $l = 1, 2, ...p$. Then for all $i \geq n$ and all j in N,

$$d(x_i, x_{i+j}) \leq (m/p) \sum_{s=0}^{p-1} \epsilon_{n+s} + \sum_{s=0}^{p-1} \epsilon_{n+s} - (q/p) \sum_{s=0}^{p-1} \epsilon_{n+s} + \epsilon_m \tag{3}$$

where $m = rp + q$ and r, p are positive integers.
Further if

$$d(x_n, y_n) \longrightarrow 0 \tag{4}$$

uniformly for all x_0, y_0 in X with $d(x_0, y_0) \leq c$. Then

$$\text{the sequence} \quad \{x_n\} \quad \text{is uniformly Cauchy.} \tag{5}$$

If the graphs of $(f_l, X, d) \quad l = 1, 2, \cdots p$ are complete and (4) holds then for $l = 1, 2, \cdots p$, $d(x, f_l) \leq c$ implies that f_l have a common fixed point.

Proof : By the induction process on k we shall prove (3) for $j \leq km$ for all k in N under the given condition that, for a given m, n,

$$(m/p)\sum_{s=0}^{p-1} \epsilon_{n+s} + \sum_{s=0}^{p-1} \epsilon_{n+s} - (q/p)\sum_{s=0}^{p-1} \epsilon_{n+s} + \epsilon_m \leq c$$

and $d(x, f_l(x)) \leq c$ for $l = 1, 2, ...p$ for all x in X.
Let $(x)_i = x_i$ defined by (1). Let $k = 1$, that is $j \leq m$ and m is a multiple of k; then for all $i \geq n$ and $j \leq m$ (2) implies that

$$d(x_i, x_{i+j}) \leq d(x_i, x_{i+p}) + d(x_{i+p}, x_{i+2p}) + \cdots + d(x_{i+tp}, x_{i+j}) \quad \text{if}(j = tp + g)$$

As $j \leq m$ implies that $t \leq r$, where $m = rp$. Then adding some more terms we get,

$$d(x_i, x_{i+j}) \leq d(x_i, x_{i+p}) + d(x_{i+p}, x_{i+2p}) + \cdots + d(x_{i+(r-1)p}, x_{i+rp})$$

Now

$$\begin{aligned} d(x_i, x_{i+j}) &\leq d((x_0)_i, (x_1)_i) + d((x_0)_{i+1}, (x_1)_{i+1}) + \cdots + d((x_0)_{i+p-1}, (x_1)_{i+p-1}) \\ &\leq \epsilon_n + \epsilon_{n+1} \cdots + \epsilon_{n+p-1} \\ &\leq \sum_{s=0}^{p-1} \epsilon_{n+s} \end{aligned}$$

So $\quad d(x_i, x_{i+j}) \leq r \sum_{s=0}^{p-1} \epsilon_{n+s}$.
That is

$$d(x_i, x_{i+j}) \leq (m/p)\sum_{s=0}^{p-1} \epsilon_{n+s} \tag{6}$$

If m is not divisible by p then $m = rp + q$. As $j \leq m$, $j = tp + g$ where $t \leq r$

$$
\begin{aligned}
& d(x_i, x_{i+j}) \\
& \leq d(x_i, x_{i+p}) + d(x_{i+p}, x_{i+2p}) + \cdots + d(x_{i+tp}, x_{i+j}) \\
& \leq d(x_i, x_{i+p}) + d(x_{i+p}, x_{i+2p}) + \cdots + d(x_{i+rp}, x_{i+m}) \\
& \leq r(\epsilon_n + \epsilon_{n+1} + \cdots + \epsilon_{n+p-1}) + d(x_{i+rp}, x_{i+rp+1}) + \cdots + d(x_{i+rp+q-1}, x_{i+rp+q}) \\
& = r \sum_{s=0}^{p-1} \epsilon_{n+s} + d((x_0)_{i+rp}, (x_1)_{i+rp}) + \cdots + d((x_0)_{i+rp+q-1}, (x_1)i + rp + q - 1) \\
& = r \sum_{s=0}^{p-1} \epsilon_{n+s} + \sum_{s=0}^{q-1} \epsilon_{n+s} \\
& = ((m-q)/p) \sum_{s=0}^{p-1} \epsilon_{n+s} + \sum_{s=0}^{q-1} \epsilon_{n+s} \\
& = (m/p) \sum_{s=0}^{p-1} \epsilon_{n+s} + \sum_{s=0}^{q-1} \epsilon_{n+s} - (q/p) \sum_{s=0}^{p-1} \epsilon_{n+s}
\end{aligned}
$$

Thus for all $i \geq n$ and $j \leq m$ independent of the condition whether m is devisible by p or not, we have

$$
d(x_i, x_{i+j}) \leq (m/p) \sum_{s=0}^{p-1} \epsilon_{n+s} + (1 - q/p) \sum_{s=0}^{p-1} \epsilon_{n+s} \tag{7}
$$

That is (3) holds for all $j \leq m$. Now we suppose that (3) holds for all $j \leq km$ and prove it for $j \leq (k+1)m$.
Consider $km \leq j \leq (k+1)m$. Then $0 < j - m \leq km$ and the induction process gives

$$
d(x_i, x_{i+j-m}) \leq (m/p) \sum_{s=0}^{p-1} \epsilon_{n+s} + (1 - q/p) \sum_{s=0}^{p-1} \epsilon_{n+s} + \epsilon_m \leq c
$$

for all $i \geq n$. Taking $x_0 = x_i$ and $y_0 = x_{i+j-m}$ and iterating above inequality m times, we get

$$
d(x_{i+m}, x_{i+j}) \leq \epsilon_m \qquad \text{for all} \quad i \geq n \tag{8}
$$

Therefore from (7) with $j = m$ and (8), we have for all $i \geq n$

$$
\begin{aligned}
d(x_i, x_{i+j}) & \leq d(x_i, x_{i+m}) + d(x_{i+m}, x_{i+j}) \\
& \leq (m/p) \sum_{s=0}^{p-1} \epsilon_{n+s} + (1 - q/p) \sum_{s=0}^{p-1} \epsilon_{n+s} + \epsilon_m
\end{aligned}
$$

Thus (3) holds for all $j \leq (k+1)m$ and hence for all j in N.
Now from (2) and (4) we have $\epsilon_n \downarrow 0$. Therefore for a given $0 < \epsilon < c$ we can choose m large enough so that $\epsilon_m < \epsilon$. Further we take n large enough and choosing $\sum_{s=0}^{p-1} \epsilon_{n+s} < p(\epsilon - \epsilon_m)/(m+p-q)$, we get

$$\sum_{s=0}^{p-1} \epsilon_{n+s} + ((p-q)/m) \sum_{s=0}^{p-1} \epsilon_{n+s} < (p/m)(\epsilon - \epsilon_m) \tag{9}$$

Thus from (9), we have

$$(m/p) \sum_{s=0}^{p-1} \epsilon_{n+s} + (1 - q/p) \sum_{s=0}^{p-1} \epsilon_{n+s} < (\epsilon - \epsilon_m)$$

i.e;

$$(m/p) \sum_{s=0}^{p-1} \epsilon_{n+s} + (1 - q/p) \sum_{s=0}^{p-1} \epsilon_{n+s} + \epsilon_m < \epsilon < c$$

So (3) holds for all j in N and hence $d(x_i, x_{i+j}) < \epsilon$ for all $i \geq n$ and j in N and all x with $d(x, f_l(x)) \leq c$ which proves (5).
Further let the graphs of (f_l, X, d), $l = 1, 2, ...p$ be complete and (4) holds for all x with $(f_l, X, d) \leq c$, $l = 1, 2, ...p$, (5) gives that $f_l(x_{n-1})$ are Cauchy for all $l = 1, 2, ...p$ and hence by graph completeness we have

$$f_l(x_{n-1}) \longrightarrow f_l(x_{n-1}) \quad \text{with} \quad x_n \longrightarrow z$$

Therefore $f_l z = z$, for $l = 1, 2, ...p$.
i.e;

$$f_1 z = f_2 z = \cdots = f_p z = z.$$

This completes the proof.

Let (f_l, X, d), $l = 1, 2, \cdots p$ be p operators on a metric space (X, d).
Define $F = f_1 f_2 \cdots f_p$. So F is an operator on (X, d). Now we state a generalised version of Theorem 1 of Leader [1] applicable for the joint operator F.

Theorem 2 : Define

$$\epsilon_n = \text{Sup}\{(F^i x, F^i y) : i \geq n, \quad d(x, y) \leq c\} \tag{10}$$

If $m\epsilon_n + \epsilon_m \leq c$ and $d(x, Fx) \leq c$ then for all $i \geq n$ and all j in N, we have

$$d(F^i x, F^{i+j} x) \leq m\epsilon_n + \epsilon_m \tag{11}$$

Hence if

$$d(F^n x, F^n y) \longrightarrow 0 \quad \text{uniformly for all } x,\, y \text{ with} \quad d(x, y) \leq c \tag{12}$$

Then

$$\text{the orbit} < F^n x > \text{with } d(x, Fx) \leq c \text{ is uniformly Cauchy.} \tag{13}$$

If the grapth of (F, X, d) is complete and (12) holds then $d(x, Fx) \leq c$ implies that the orbit of x converges to a fixed point $z = Fz$ and for ϵ_n defined by (10), we have

$$d(F^n x, z) \leq m\epsilon_n + \epsilon_m \quad if \quad m\epsilon_n + \epsilon_m \leq c \tag{14}$$

If $d(Fx, Fy) \leq d(x, y)$ then F has a unique fixed point. Further if $f_i f_j(x) = f_j f_i(x)$ for all $1 \leq i, j \leq p$ and all x in X then the fixed point of F becomes the unique coomon fixed point of $f_1, f_2, \cdots, f_p$.

The proof goes in the similar fashion as that of Theorem 2 of Som et al [2], so we omit it .

Definition : Let $f_1, f_0 : (X, d) \longrightarrow (X, d)$ be two maps with $f_0(X) \subset f_1(X)$. If there exists a point $t \in X$ such that $f_0(t) = f_1(t)$ then we say that f_0 has an f_1-generalized fixed point.

Theorem 3 : Let (X, d) be a metric space. Let $f_0, f_1, \cdots, f_p$ be $(p + 1)$ surjective mappings of X into X with f_r continuous for all $1 \leq r \leq p$. Let $f_r f_s = f_s f_r$ for all $0 \leq r, s \leq p$. Now, taking x_0 in X define sequence $\{x_n\}$ as :

$$\begin{aligned} x_{n+1} &= f_n(x_n) \qquad \text{if} \quad n + 1 \leq p + 1 \\ &= f_{q-1}(x_n) \quad \text{if} \quad n + 1 = t(p + 1) + q \\ &= f_p(x_n) \qquad \text{if} \quad n + 1 = t(p + 1) \end{aligned}$$

Similarly define $\{y_n\}$ with initial point y_0. For some $c \geq 0$, define

$$\epsilon_n = \text{Sup}\{d(x_i, y_i) : i \geq n - 1, \quad d(f_0(x_0), f_0(y_0)) \leq c\} \tag{15}$$

If

$$(m/(p + 1)) \sum_{s=0}^{p} \epsilon_{n+s} + \sum_{s=0}^{p} \epsilon_{n+s} - (q/(p + 1)) \sum_{s=0}^{p} \epsilon_{n+s} + \epsilon_m \leq c$$

and

$$d(f_r(x), f_s(x)) \leq c, \quad \text{for } 0 \leq r, s \leq p$$

Then for all $i \geq n-1$ and all j in N,

$$d(x_i, x_{i+j}) \leq (m/(p+1)) \sum_{s=0}^{p} \epsilon_{n+s} + \sum_{s=0}^{p} \epsilon_{n+s} - (q/(p+1)) \sum_{s=0}^{p} \epsilon_{n+s} + \epsilon_m \quad (16)$$

where $m = r(p+1) + q$ and r, p are positive integers.
Hence if

$$d(x_n, y_n) \longrightarrow 0 \quad (17)$$

uniformly for all $x_0, y_0 \in X$ with $d(f_0(x_0), f_0(y_0)) \leq c$ then the sequence $\{x_n\}$ is uniformly Cauchy, and further if $f_r, \quad 0 \leq r \leq p$ satisfies :

$$d(f_0(x), f_0(y)) \leq d(f_1(x), f_1(y)) \cdots \leq d(f_p(x), f_p(y)) \quad \text{for all } x, y \in X \quad (18)$$

then f_r have a f_s-generalized fixed point for all $0 \leq r < s \leq p$.

Proof : The proof of (16) and that $\{x_n\}$ is uniformly Cauchy goes in a similar fashion as that of Theorem 1, so we omit it.
Since f_r is continuous, we have $f_r(x_n) \to f_r(t)$ as $\{x_n\} \to t$ for all $1 \leq r \leq p$.
Now (18) gives

$$d(f_0(x_n), f_0(t)) \leq d(f_1(x_n), f_1(t)) \cdots \leq d(f_p(x_n), f_p(t))$$

This implies $f_0(x_n) \to f_0(t)$ as $\{x_n\} \to t$. Consider the subsequence $\{x_{m(p+1)}\}$ of $\{x_n\}$. By definition we can write $\{x_{m(p+1)}\} = (f_p f_{p-1} \cdots f_1 f_0)^m (x_0)$.
Now if we choose n such that $(n+1) = m(p+1) + 1$, for some m then for any $1 \leq r \leq p$ we have

$$\begin{aligned}
f_r(x_{n+1}) &= f_r(f_0(x_n)) \\
&= f_r(f_0(x_{m(p+1)})) \\
&= f_r(f_0(f_p f_{p-1} \cdots f_1 f_0)^m (x_0))) \\
&= f_0(f_r(f_p f_{p-1} \cdots f_1 f_0)^m (x_0))) \\
&= f_0(f_p^m \cdots f_r^{m+1} \cdots f_0^m)(x_0) \\
&\longrightarrow f_0(t) \quad \text{as} \quad m \longrightarrow \infty
\end{aligned}$$

Therefore $f_r(t) = f_0(t)$, for all $1 \leq r \leq p$. So $f_0(t) = f_1(t) \cdots = f_p(t)$. Hence f_r have a f_s-generalized fixed point for all $0 \leq r < s \leq p$.

Theorem 4 : Let (f_l, X, d), $l = 1, 2, \cdots, p$ be p operators with complete graphs. Let (X,d) be weakly c-chained and (4) holds. Then f_l have a common fixed point.

Proof : Let x_n and y_n be two sequences as defined in (1) with initial points x_0 and y_0 respectively. Since (X,d) is weakly c-chained, we have for any $x,y \in X$ a finite sequence $< x^0, x^1, \cdots x^m >$ with $x^0 = x, x^m = y$ and $d(x^i, x^i + 1) \leq c$ for $i = 0,1,\cdots,m-1$
Then in x_n, y_n applying triangle inequality, we have

$$\begin{aligned} d(x_n,y_n) &= d(x_n^0, x_n^m) \\ &\leq d(x_n^0, x_n^1) + d(x_n^1, x_n^2) + \cdots d(x_n^{m-1}, x_n^m) \end{aligned} \tag{19}$$

where $d(x_n^i, x_n^{i+1}) \leq c$ for $i = 0,1,2,\cdots,m-1$
By using similar arguments as in (6) and (7) in the above and applying (4) we get $d(x_n,y_n) \longrightarrow 0$ as $n \rightarrow 0$ for all $x,y \in X$.
In case $y_n = f_l(x_n)$

$d(x_n, f_l(x_n)) \longrightarrow 0$ if $n+1 = tp+l$.
Therefore for n large enough we have $d(x_n, f_l(x_n)) \leq c$ for $l = 1,2,\cdots,p$
Hence Theorem 1 gives the result.

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References

[1] S.Leader,"Fixed Points for operators on metric spaces with conditional uniform equivalence of orbits",J.Math. Anal. Appl. 61 (1977), 466-477.

[2] Tanmoy Som And R.N. Mukherjee, " Generalization of Leader's Fixed Point Principle", Bull. Austral. Math. Soc. 29 (1984) , 349-356.

[3] A. Meir and E. Keeler, "A theorem on contraction mappings", J. Math. Anal. Appl. 28 (1969), 326-329.

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