An Integral Representation of Some $k$-Hypergeometric Functions

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Abstract
In this paper, we introduce a new and simple integral representation of some $k$-confluent hypergeometric functions and $k$-hypergeometric functions.

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1 Introduction
Diaz and Pariguan [1] have introduced and proved some identities of $k$-gamma function, $k$-beta function and $k$-Pochhammer symbol. They have deduced an integral representation of $k$-gamma function, $k$-beta function respectively given by

$$\Gamma_k(x) = k^x \Gamma\left(\frac{x}{k}\right) = \int_0^\infty t^{x-1} e^{-t} dt, \quad \text{Re}(x) > 0, \quad k > 0 \tag{1.1}$$

and

$$B_k(x, y) = \frac{1}{k} \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x > 0, \quad y > 0. \tag{1.2}$$
They have also provided some useful and applicable relations

\[ B_k(x, y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, \quad (1.3) \]

\[ (x)_{j,k} = \frac{\Gamma_k(x+jk)}{\Gamma_k(x)}; \quad (1.4) \]

where \((x)_{j,k} = x(x+k)(x+2k)...(x+(j-1)k)\), is the \(k\)-Pochhammer symbol

and \(\sum_{j=0}^{\infty} (\alpha)_{j,k} \frac{x^j}{j!} = (1-kx)^{-\frac{\alpha}{k}}. \quad (1.5)\)

Recently, Mansour [5] and Kokologiannaki [4] have proved a number of properties and Kokologiannaki has also taken up \(k\)-Zeta function

\[ \zeta(x,s) = \sum_{j=0}^{\infty} \frac{1}{(x+jk)^s}, k, x > 0, s > 1. \quad (1.6) \]

The main purpose of this paper is to introduce an integral representation of some \(k\)-confluent hypergeometric functions and \(k\)-hypergeometric functions so that we can get the usual integral representations discussed in [2, 3], by taking \(k \to 1\). We shall use later the following basic results.

\[ m^{mj}\left(\frac{x}{m}\right)_{j,k} \left(\frac{x+jk}{m}\right)_{j,k} \cdots \left(\frac{x+(m-1)k}{m}\right)_{j,k} = (x)_{m,j,k}; \quad (1.7) \]

\[ (x)_{mj,k} = \frac{\Gamma_k(x+mjk)}{\Gamma_k(x)}; \quad (1.8) \]

\[ \sum_{j=0}^{\infty} \frac{x^j}{j!} = e^x. \quad (1.9) \]

## 2 Integral Representation of Some \(k\)-Confluent Hypergeometric Functions

In this section, we determine integral representations of some \(k\)-confluent hypergeometric functions \(mF_{m,k}\).

**Theorem 2.1:**

If \(\text{Re}(\gamma) > \text{Re}(\beta) > 0\), \(k > 0\), \(m \geq 1\), \(m \in \mathbb{Z}^+\), then for all finite \(x\)

\[ mF_{m,k}\left(\frac{\beta}{m},k;\frac{\beta+k}{m},k;\ldots;\frac{\gamma+(m-1)k}{m},k;\right) = x \]

\[ \left(\frac{\gamma}{m},k,\frac{\gamma+k}{m},k,\ldots;\frac{\gamma+(m-1)k}{m},k;\right) \]
\[ k - \text{hypergeometric functions} \]

\[ \int_0^1 t^{k-1} (1-t)^{\frac{p-\beta-1}{k}} e^{xt} \text{ } dt. \quad (2.1) \]

**Proof:** First note that for any positive integer \( j \), we get

\[
\frac{(\beta)_{mj,k}}{(\gamma)_{mj,k}} = \frac{\Gamma_k(\gamma) \Gamma_k(\beta + mj_k)}{\Gamma_k(\beta) \Gamma_k(\gamma + mj_k)} = \frac{\Gamma_k(\gamma)}{\Gamma_k(\beta) \Gamma_k(\gamma - \beta)} B_k(\beta + mj_k, \gamma - \beta)
\]

\[
= \frac{\Gamma_k(\gamma)}{k \Gamma_k(\beta) \Gamma_k(\gamma - \beta)} \int_0^1 t^{\frac{p+mj-1}{k}} (1-t)^{\frac{p-\beta-1}{k}} \text{ } dt. \quad (2.2)
\]

Now, using Equations (1.7), (1.9) and (2.2), we get

\[
m_{F_{m,k}} \left( \left( \frac{\beta}{m}, k \right), \left( \frac{\beta+k}{m}, k \right), \ldots, \left( \frac{\gamma+(m-1)k}{m}, k \right) \right) = \sum_{j=0}^{\infty} \frac{(\beta)_{mj,k} x^j}{j!} = \frac{\Gamma_k(\gamma)}{k \Gamma_k(\beta) \Gamma_k(\gamma - \beta)} \int_0^1 t^{\frac{p-1}{k}} (1-t)^{\frac{p-\beta-1}{k}} e^{xt} \text{ } dt.
\]

**Corollary 2.2:**

If \( \text{Re}(\gamma) > \text{Re}(\beta) > 0 \), then for all finite \( x \)

\[
_1 F_{1,k} \left( (\beta,k);(\gamma,k);x \right) = \frac{\Gamma_k(\gamma)}{k \Gamma_k(\beta) \Gamma_k(\gamma - \beta)} \int_0^1 t^{\frac{p-1}{k}} (1-t)^{\frac{p-\beta-1}{k}} e^{xt} \text{ } dt. \quad (2.3)
\]
3 Integral Representation of Some $k$-Hypergeometric Functions

In this section, we determine integral representations of some $k$-hypergeometric functions.

**Theorem 3.1:**

If $\Re(\gamma) > \Re(\beta) > 0$, $k > 0$, $m \geq 1$, $m \in \mathbb{Z}^+$ and $|x| < 1$, then

$$
\begin{aligned}
F_{m,k}^{m+1} & \left( \begin{array}{c}
(\alpha, k), \left( \frac{\beta}{m}, k \right), \left( \frac{\beta+k}{m}, k \right), \ldots, \left( \frac{\gamma+(m-1)k}{m}, k \right) \\
(\frac{\gamma}{m}, k), \left( \frac{\gamma+k}{m}, k \right), \ldots, \left( \frac{\gamma+(m-1)k}{m}, k \right)
\end{array} \right) ; x \\
\end{aligned}
= \frac{\Gamma_k(\gamma)}{k \Gamma_k(\beta) \Gamma_k(\gamma - \beta)} \int_0^1 \frac{t^{\beta-1}}{1-t} \left( \frac{\gamma - \beta}{k} \right) \left( 1 - kxt^m \right)^{-\frac{\alpha}{k}} dt.
$$

(3.1)

**Proof:** First note that for any positive integer $j$, we have

$$
\frac{(\beta)_{mj,k}}{(\gamma)_{mj,k}} = \frac{\Gamma_k(\gamma) \Gamma_k(\beta + mjk)}{\Gamma_k(\beta) \Gamma_k(\gamma + mjk)}
= \frac{\Gamma_k(\gamma)}{\Gamma_k(\beta) \Gamma_k(\gamma - \beta)} B_k(\beta + mjk, \gamma - \beta)
= \frac{\Gamma_k(\gamma)}{k \Gamma_k(\beta) \Gamma_k(\gamma - \beta)} \int_0^{\beta + mj - 1} (1 - t)^{\gamma - \beta - 1} dt.
$$

(3.2)

Using Equations (1.7), (3.2) and (1.5), we get

$$
\begin{aligned}
F_{m,k}^{m+1} & \left( \begin{array}{c}
(\alpha, k), \left( \frac{\beta}{m}, k \right), \left( \frac{\beta+k}{m}, k \right), \ldots, \left( \frac{\gamma+(m-1)k}{m}, k \right) \\
(\frac{\gamma}{m}, k), \left( \frac{\gamma+k}{m}, k \right), \ldots, \left( \frac{\gamma+(m-1)k}{m}, k \right)
\end{array} \right) ; x \\
= \sum_{j=0}^{\infty} \frac{(\alpha)_{j,k} (\beta)_{mj,k}}{(\gamma)_{mj,k}} \frac{x^j}{j!}
\end{aligned}
$$

$$
= \sum_{j=0}^{\infty} \frac{(\alpha)_{j,k} (\beta)_{mj,k}}{(\gamma)_{mj,k}} \frac{x^j}{j!}
$$
\[ k - \text{hypergeometric functions} \quad 207 \]

\[ \frac{\Gamma_k\left(\gamma\right)}{k \Gamma_k\left(\beta\right) \Gamma_k\left(\gamma - \beta\right)} \int_0^1 t^{k-1} \left(1 - t\right)^{-\frac{\beta}{k}} \left(1 - kxt^m\right)^{-\frac{a}{x}} \, dt. \]

**Corollary 3.1:**

If \( \text{Re}(\gamma) > \text{Re}(\beta) > 0 \) and \( |x| < 1 \), then

\[ _2 F_{1,k} \left( (\alpha, k), (\beta, k); (\gamma, k); x \right) \]

\[ = \frac{\Gamma_k\left(\gamma\right)}{k \Gamma_k\left(\beta\right) \Gamma_k\left(\gamma - \beta\right)} \int_0^1 t^{k-1} \left(1 - t\right)^{-\frac{\beta}{k}} \left(1 - kxt^m\right)^{-\frac{a}{x}} \, dt. \quad (3.3) \]

**References**


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