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Equivalence between the Category of Cat¹-Complexes and the Category of Crossed Modules of Complexes

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Abstract. In this paper, we introduce the notion of cat¹-complexes as a suitable generalization of the cat¹-groups through embedding the category of cat¹-groups in the category of Cat¹-complexes as a subcategory (via isomorphism of categories). It was shown that the category of Cat¹-complexes is equivalent to the category of crossed modules of complexes as well as the couple of two covariant functors used to show the equivalence between these two categories are represent an adjoint pair of functors.

Keywords. (chain) complex; Cat¹-complex; crossed module; Adjoint functors

1. Introduction

Crossed modules of groups were originally introduced by J.H.C. Whitehead [6] 1949. A crossed module of groups (c.f. [1] and [2] for a more detailed treatment) (C, G, ∂) is a group homomorphism $\partial: C \rightarrow G$ together with an action of G on the left of C satisfying $(CM1)$ $\partial({}^{g}c)=g\partial(c)g^{-1}$ and $(CM2)$ ${}^{\partial(c_1)}c_2 = c_1c_2c_1^{-1}$ for all $c, c_1, c_2 \in C$ and $g \in G$. A morphism of crossed modules of groups (μ, η):$(C, G, \partial) \rightarrow (D, H, \delta)$ is a pair of group homomorphisms $\mu: C \rightarrow D$ and $\eta: G \rightarrow H$ such that $\delta\mu = \eta\partial$ and $\mu({}^{g}c) = {}^{\eta(g)}\mu(c)$ for all $c \in C$ and $g \in G$. Crossed modules of groups and morphisms as defined above form a category, $CModGrps$.

Loday [4] defines cat¹-groups and showed that the category of crossed modules of groups is equivalent to the category of cat¹-groups. Recall that a cat¹-group (G, s, t) consists of a group G and two endomorphisms $s, t: G \rightarrow G$ satisfying $(CAT1)$ $ts = s$, $st = t$ and $(CAT2)$ $[kers, kert] = \{1_G\}$. A morphism of cat¹-groups $f: (G, s, t) \rightarrow (G', s', t')$ is a group homomorphism $f: G \rightarrow G'$ such that $s'f = fs$, $t'f = ft$. Cat¹-groups and morphisms as defined above form a category, $Cat^1\text{-}Grps$.

The main aim of this paper is to extend the notion of cat[1]-groups by replacing complexes instead of groups and introduce the notion of Cat[1]-complexes. Since the group involved in the definition of a cat[1]-group is in general nonabelian, we shall assume throughout this paper that the groups which are involved in the construction of a complex not required to be abelian groups, i.e. a complex (G_*, η_*) is a sequence of groups and homomorphisms

$$--- \longrightarrow G_{n+1} \xrightarrow{\eta_{n+1}} G_n \xrightarrow{\eta_n} G_{n-1} \longrightarrow ---$$

such that $\eta_n\, \eta_{n+1}$=0 (i.e. $Im\eta_{n+1} \subseteq Ker\eta_n$) for all $n \in Z$. If G_n is an abelian group for all $n \in Z$, we shall call (G_*, η_*) an abelian complex. In this case an abelian complex is precisely a chain complex (as in the literature), for more information on chain complexes we refer the reader to [5].

We call (G'_*, η'_*) a subcomplex of a complex (G_*, η_*) if G'_n is a subgroup of G_n and $\eta'_n = \eta_n | G'_n$ is the restriction of η_n on G'_n for all $n \in Z$. A subcomplex (G'_*, η'_*) of a complex (G_*, η_*) is called normal subcomplex if G'_n is a normal subgroup of G_n all $n \in Z$. Let (C_*, μ_*) and (G_*, η_*) be complexes if $k_n = C_n \times G_n$ and $\gamma_n = \mu_n \times \eta_n$ for all $n \in Z$, therefore (K_*, γ_*) is a complex, which we call the direct product of complexes (C_*, μ_*) and (G_*, η_*) and which denotes by $(C_* \times G_*, \mu_* \times \eta_*)$.

Kamil [3] generalized the direct product of complexes, as defined above, as follows. Let G_n has a left action on C_n ($\forall n \in Z$), then we can form the semidirect product of groups, $C_n \rtimes G_n$, which is a group under the binary operation defined by $(c_n, g_n)(c'_n, g'_n) = (c_n\, {}^{g_n}c'_n, g_n g'_n)$ for all $(c_n, g_n), (c'_n, g'_n) \in C_n \rtimes G_n$. We should remark here that $\mu_n \rtimes \eta_n : C_n \rtimes G_n \to C_{n-1} \rtimes G_{n-1}$ which is defined by $(\mu_n \rtimes \eta_n)(c_n, g_n) = (\mu_n(c_n), \eta_n(g_n))$ for all $c_n \in C_n$, $g_n \in G_n$ is not necessarily a group homomorphism, while as each of (C_*, μ_*) and (G_*, η_*) is a complex, we deduce that $(\mu_n \rtimes \eta_n)(\mu_{n+1} \rtimes \eta_{n+1}) = \mu_n \mu_{n+1} \rtimes \eta_n \eta_{n+1} = 0$. Kamil [3] gave a sufficient and necessary condition for which $(\mu_n \rtimes \eta_n)$ becomes a group homomorphism, and he defined the semidirect product of complexes as follow. Let (C_*, μ_*) and (G_*, η_*) be complexes such that G_n has a left action on C_n ($\forall n \in Z$). The semidirect product of (C_*, μ_*) and (G_*, η_*), denoted by $(C_*, \mu_*) \rtimes (G_*, \eta_*)$, is defined to be the complex $(C_* \rtimes G_*, \mu_* \rtimes \eta_*)$,

$$--- \longrightarrow C_{n+1} \rtimes G_{n+1} \xrightarrow{\mu_{n+1} \rtimes \eta_{n+1}} C_n \rtimes G_n \xrightarrow{\mu_n \rtimes \eta_n} C_{n-1} \rtimes G_{n-1} \longrightarrow ---$$

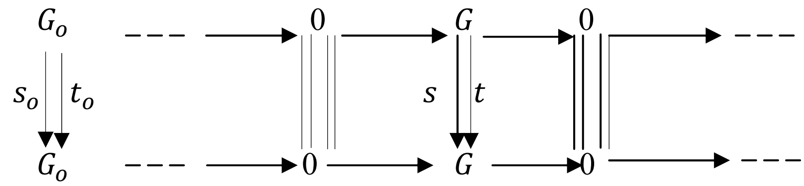
If, and only if, $\mu_n({}^{g_n}c_n) = {}^{\eta_n(g_n)}\mu_n(c_n)$ for all $c_n \in C_n$, $g_n \in G_n$. (#)

The idea of extension from cat[1]-groups to cat[1]-complexes is given as follow. It is obvious that each group G can be viewed as a length zero complex;

$$G_o \qquad --- \longrightarrow 0 \longrightarrow G \longrightarrow 0 \longrightarrow ---$$

and vice-versa. This shows that the category of groups, $Grps$, is isomorphic to the category of length zero complexes, $Comp^{(0)}$, i.e. $Grps \approx Comp^{(0)}$. Accordingly, each Cat[1]-group (G, s, t)

can also be viewed as a pair of two chain maps $s_o, t_o: G_o \to G_o$ from a length zero chain complex G_o into itself,



(⁎)

Which we shall call it in this paper a cat¹- length zero complex (G_o, s_o, t_o) (in the sense that each pair of vertical homomorphisms of (⁎) is a cat¹- group and each homomorphism in the top arrow (or in the bottom arrow) of (⁎) is a morphism of cat¹- groups (i.e. all cat¹-group information are encoded in the above diagram). In this case, the category of cat¹-groups is isomorphic to the category of cat¹-length zero complexes, Cat¹- $Comp^{(0)}$, i.e. $Cat^1\text{-}Grps \approx$ Cat¹- $Comp^{(0)}$.

Kamil [3] extended the definition of a crossed module of groups by replacing complexes instead of groups and introduced the concept of a crossed module of complexes.

Recall that a crossed module of complexes $\big((C_*,\mu_*),(G_*,\eta_*),\partial=\{\partial_n\}\big)$ is a chain map $\partial: C_* \to G_*$ such that $\partial_n: C_n \to G_n$ is a crossed module of groups and $\mu_n({}^{g_n}c_n) = {}^{\eta_n(g_n)}\mu_n(c_n)$ for all $n \in Z$. A morphism of a crossed modules of complexes $(f=\{f_n\}, l=\{l_n\}): \big((C_*,\mu_*),(G_*,\eta_*),\partial=\{\partial_n\}\big) \to \big((C_*^{'},\mu_*^{'}),(G_*^{'},\eta_*^{'}),\partial^{'}=\{\partial_n^{'}\}\big)$ is a pair of chain maps $f: C_* \to C_*^{'}$ and $l: G_* \to G_*^{'}$ such that $(f_n, l_n): (C_n, G_n, \partial_n) \to (C_n^{'}, G_n^{'}, \partial_n^{'})$ is a morphism of crossed modules of groups for all $n \in Z$. Crossed modules of complexes and morphisms as defined above form a category, $CModComp$.

In the next section, we introduce a suitable generalization of cat¹-groups namely cat¹-complexes through embedding the category of cat¹-groups in the category of cat¹- complexes, $Cat^1\text{-}Comp$. In this paper we will show that the two categories $CModComp$ and $Cat^1\text{-}Comp$ are equivalent.

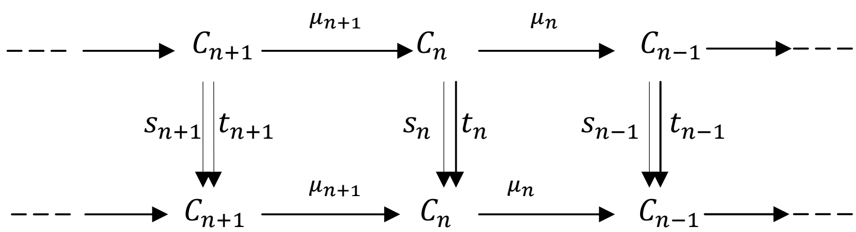
2. Cat¹-complexes

DEFINITION 2.1. A cat¹- complex is a triple $\big((C_*,\mu_*), s=\{s_n\}, t=\{t_n\}\big)$ such that (C_*,μ_*) is a complex and $s,t: C_* \to C_*$ are chain maps satisfying

(i) $ts = s$, $st = t$

(ii) $[Kers_n, Kert_n] = \{1_{G_n}\}$ for all $n \in Z$

Here is the picture of a cat¹- complexes in unabbreviated form.



In other words $((C_*,\mu_*), s=\{s_n\}, t=\{t_n\})$ is called cat¹-complex if (C_n, s_n, t_n) is a cat¹-group and $\mu_n: (C_n, s_n, t_n) \to (C_{n-1}, s_{n-1}, t_{n-1})$ is a morphism of cat¹- groups for all $n \in Z$.

DEFINITION 2.2. A morphism $f=\{f_n\}: ((C_*,\mu_*), s=\{s_n\}, t=\{t_n\}) \to ((G_*,\eta_*), u=\{u_n\}, v=\{v_n\})$ of cat¹-complexes is a chain map $f=\{f_n\}: (C_*,\mu_*) \to (G_*,\eta_*)$ such that $f_n: (C_n, s_n, t_n) \to (G_n, u_n, v_n)$ is a morphism of cat¹-groups for all $n \in Z$.

It is clear that if $I_{C_*}: (C_*,\mu_*) \to (C_*,\mu_*)$ is the identity chain map on (C_*,μ_*) then $I_{(C_*,s,t)}=I_{C_*}: ((C_*,\mu_*), s=\{s_n\}, t=\{t_n\}) \to ((C_*,\mu_*), s=\{s_n\}, t=\{t_n\})$ is a morphism of cat¹-complexes. Also, if $f=\{f_n\}: ((C_*,\mu_*), s, t) \to ((C'_*,\mu'_*), s', t')$ and $l=\{l_n\}: ((C'_*,\mu'_*), s', t') \to ((C''_*,\mu''_*), s'', t'')$ are morphisms of cat¹-complexes, then their composition $lf=\{l_n f_n\}: ((C_*,\mu_*), s, t) \to ((C''_*,\mu''_*), s'', t'')$ is a morphism of cat¹- complexes.

Taking objects and morphisms as defined above, we obtain the category $Cat^1\text{-}Comp$ of cat¹-complexes. Note that, $Cat^1\text{-}Comp^{(0)} \subseteq Cat^1\text{-}Comp$, and since $Cat^1\text{-}Grps \approx Cat^1\text{-}Comp^{(0)}$, we deduce that $Cat^1\text{-}Grps \subseteq Cat^1\text{-}Comp$, i.e. the category of cat¹-groups is embedding in the category of cat¹- complexes (via isomorphism of categories).

EXAMPLES 2.3.. (1) Any complex (C_*,μ_*) may be regarded as a cat¹-complex $((C_*,\mu_*), I_{C_*}=\{I_{C_n}\}, I_{C_*}=\{I_{C_n}\})$. Accordingly $Comp$ is a full subcategory of $Cat^1\text{-}Comp$.

(3) Let $(\mathbb{Z},+)$ be the additive group of integers and (C_*,μ_*) , (G_*,η_*) be two complexes defined as follows.

$$C_n=\mathbb{Z}=G_n,\ \mu_{n+m}=\begin{cases}0 & if\ m\ is\ even\\ f_2 & if\ m\ is\ odd\end{cases} \quad \text{and} \quad \eta_{n+m}=\begin{cases}f_2 & if\ m\ is\ even\\ 0 & if\ m\ is\ odd\end{cases},$$

where $f_2: \mathbb{Z} \to \mathbb{Z}$ is a group homomorphism defined by $f_2(x) = 2x$. The following commutative diagram thus represents a cat¹-complex.

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{0\times f_2} & \mathbb{Z}\times\mathbb{Z} & \xrightarrow{f_2\times 0} & \mathbb{Z}\times\mathbb{Z} & \xrightarrow{0\times f_2} & \mathbb{Z}\times\mathbb{Z} & \xrightarrow{f_2\times 0} & \mathbb{Z}\times\mathbb{Z} & \longrightarrow \cdots \\
 & u \downarrow\downarrow v & & s \downarrow\downarrow t & & u \downarrow\downarrow v & & s \downarrow\downarrow t & \\
\cdots \xrightarrow{0\times f_2} & \mathbb{Z}\times\mathbb{Z} & \xrightarrow{f_2\times 0} & \mathbb{Z}\times\mathbb{Z} & \xrightarrow{0\times f_2} & \mathbb{Z}\times\mathbb{Z} & \xrightarrow{f_2\times 0} & \mathbb{Z}\times\mathbb{Z} & \longrightarrow \cdots
\end{array}$$

Where $u, v, s, t: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ are defined as $u(x,y)=s(x,y)=t(x,y)=(0,y)$ and $v(x,y)=(0,2x+y)$ for all $(x,y) \in \mathbb{Z}\times\mathbb{Z}$.

3. Equivalence between $CModComp$ and Cat^1-$Comp$

LEMMA 3.1. There are two covariant functors;

(1) The functor T: $CModComp \to Cat^1\text{-}Comp$ is defined by:

(i) $T\big((C_*,\mu_*),(G_*,\eta_*),\partial=\{\partial_n\}\big)=\big((C_*\rtimes G_*,\mu_*\rtimes\eta_*),s=\{s_n\},t=\{t_n\}\big)$ where $s,t:(C_*\rtimes G_*,\mu_*\rtimes\eta_*)\to(C_*\rtimes G_*,\mu_*\rtimes\eta_*)$ are defined by $s_n(c_n,g_n)=(1_{C_n},g_n)$, and $t_n(c_n,g_n)=(1_{C_n},\partial_n(c_n)g_n)$ for all $(c_n,g_n)\in C_n\rtimes G_n$ and all $n\in Z$.
(ii) $T(f,l)=f\rtimes l$, For all $(f=\{f_n\},l=\{l_n\}):\big((C_*,\mu_*),(G_*,\eta_*),\partial=\{\partial_n\}\big)\longrightarrow\big((H_*,\alpha_*),(D_*,\beta_*),\lambda=\{\lambda_n\}\big)$ in $CModComp$.
(2) The functor $R:Cat^1\text{-}Comp\to CModComp$ is defined by:
(i) $R\big((C_*,\mu_*),s=\{s_n\},t=\{t_n\}\big)=\big((Kers_*,\mu_*'),(Ims_*,\mu_*''),t|Kers_*\big)$ for all $\big((C_*,\mu_*),s=\{s_n\},t=\{t_n\}\big)\in ObCat^1\text{-}Comp$, where μ_n' is the restriction $\mu_n|Kers_n$, μ_n'' is the restriction $\mu_n|Ims_n$ and Ims_n acts on $Kers_n$ by conjugations for all $n\in Z$.
(ii) $R(f)=(f',f'')$ for all $f=\{f_n\}:\big((C_*,\mu_*),s=\{s_n\},t=\{t_n\}\big)\to\big((G_*,\eta_*),u=\{u_n\},v=\{v_n\}\big)$ in $Cat^1\text{-}Comp$.

THEOREM 3.2. The two categories $Cat^1\text{-}Comp$ and $CModComp$ are equivalent.

Poof. From lemma (3.1), we need only to show that $R\circ T\approx I_{CModComp}$ and $T\circ R\approx I_{Cat^1-Comp}$. Define a function $\Phi:R\circ T\longrightarrow I_{CModComp}$ as follows:
Let $A=\big((C_*,\mu_*),(G_*,\eta_*),\partial=\{\partial_n\}\big)\in ObCModComp$,

$\Phi(A):\big((Kers_*,(\mu_*\rtimes\eta_*)'),(Ims_*,(\mu_*\rtimes\eta_*)''),t|Kers_*\big)\to\big((C_*,\mu_*),(G_*,\eta_*),\partial=\{\partial_n\}\big)$
such that $\Phi(A)=(\pi_{C_*},\pi_{G_*})$, where $\pi_{C_*}=\{\pi_{C_n}\}:(Kers_*,(\mu_*\rtimes\eta_*)')\to(C_*,\mu_*)$ and $\pi_{G_*}=\{\pi_{G_n}\}:(Ims_*,(\mu_*\rtimes\eta_*)'')\to(G_*,\eta_*)$ are chain maps defined by; $\pi_{C_n}(c_n,1_{G_n})=c_n$ and $\pi_{G_n}(1_{C_n},g_n)=g_n$, for all $(c_n,1_{G_n})\in Kers_n$, $(1_{C_n},g_n)\in Ims_n$ and all $n\in Z$. Note that (π_{C_*},π_{G_*}) is indeed a morphism of crossed modules of complexes since π_{C_*} and π_{G_*} are chain maps, $\partial\pi_{C_*}=\pi_{G_*}(t|Kers_*)$ and $\pi_{C_n}\big({}^{(1_{C_n},g_n)}(c_n,1_{G_n})\big)={}^{\pi_{G_n}(1_{C_n},g_n)}\pi_{C_n}(c_n,1_{G_n})$ for all $n\in Z$. Now, for any $(f=\{f_n\},l=\{l_n\}):A=\big((C_*,\mu_*),(G_*,\eta_*),\partial=\{\partial_n\}\big)\to B=\big((H_*,\alpha_*),(D_*,\beta_*),\lambda=\{\lambda_n\}\big)$,
$R\circ T(f,l)=((f\rtimes l)',(f\rtimes l)''):\big((Kers_*,(\mu_*\rtimes\eta_*)'),(Ims_*,(\mu_*\rtimes\eta_*)''),t|Kers_*\big)\longrightarrow\big((Keru_*,(\alpha_*\rtimes\beta_*)'),(Imv_*,(\alpha_*\rtimes\beta_*)''),v|Keru_*\big)$

To show that Φ is a natural transformation, it is enough to show the commutativity of the following diagram:

$$\begin{array}{ccc}
\big((Kers_*,(\mu_*\rtimes\eta_*)'),(Ims_*,(\mu_*\rtimes\eta_*)''),t|Kers_*\big) & \xrightarrow{\Phi(A)} & \big((C_*,\mu_*),(G_*,\eta_*),\partial\big)\\
\Big\downarrow{\scriptstyle((f\rtimes l)',(f\rtimes l)'')} & & \Big\downarrow{\scriptstyle(f,l)}\\
\big((Keru_*,(\alpha_*\rtimes\beta_*)'),(Imv_*,(\alpha_*\rtimes\beta_*)''),v|Keru_*\big) & \xrightarrow{\Phi(B)} & \big((H_*,\alpha_*),(D_*,\beta_*),\lambda\big)
\end{array}$$

Note that $(f,l)\Phi(A)=(f,l)(\pi_{C_*},\pi_{G_*})=(f\pi_{C_*},l\pi_{G_*})$. On the other hand;
$\Phi(B)((f\rtimes l)',(f\rtimes l)'')=(\pi_{H_*},\pi_{D_*})((f\rtimes l)',(f\rtimes l)'')=(\pi_{H_*}(f\rtimes l)',\pi_{D_*}(f\rtimes l)'')$.

Since $(f_n \rtimes l_n)' = f_n \rtimes (l_n | Kers_n)$ and $(f_n \rtimes l_n)'' = f_n \rtimes (l_n | Ims_n)$, therefore $f_n \pi_{C_n} = \pi_{H_n}(f_n \rtimes l_n)'$ and $l_n \pi_{G_n} = \pi_{D_n}(f_n \rtimes l_n)''$. Thus Φ is natural transformation. Now define a function $\Psi: I_{CModComp} \to R \circ T$ as follows: let $A = \big((C_*, \mu_*), (G_*, \eta_*), \partial = \{\partial_n\}\big) \in ObCModComp$,
$\Psi(A): \big((C_*, \mu_*), (G_*, \eta_*), \partial = \{\partial_n\}\big) \to \big((Kers_*, (\mu_* \rtimes \eta_*)'), (Ims_*, (\mu_* \rtimes \eta_*)''), t|Kers_*\big)$ such that $\Psi(A) = \big(\pi_{C_*}^{-1}, \pi_{G_*}^{-1}\big)$, where $\big(\pi_{C_*}^{-1} = \{\pi_{C_n}^{-1}\}, \pi_{G_*}^{-1} = \{\pi_{G_n}^{-1}\}\big)$ is a morphism of crossed modules of complexes defined by: $\pi_{C_n}^{-1}(c_n) = (c_n, 1_{G_n})$ and $\pi_{G_n}^{-1}(g_n) = (1_{C_n}, g_n)$ for all $c_n \in C_n$, $g_n \in G_n$ and all $n \in Z$. By using a similar argument as above, one can show that Ψ is also a natural transformation according to the commutativity of the following diagram;

$$\begin{array}{ccc}
\big((C_*, \mu_*), (G_*, \eta_*), \partial\big) & \xrightarrow{\ \Psi(A)\ } & \big((Kers_*, (\mu_* \rtimes \eta_*)'), (Ims_*, (\mu_* \rtimes \eta_*)''), t|Kers_*\big) \\
\Big\downarrow (f, l) & & \Big\downarrow \big((f \rtimes l)', (f \rtimes l)''\big) \\
\big((H_*, \alpha_*), (D_*, \beta_*), \lambda\big) & \xrightarrow{\ \Psi(B)\ } & \big((Keru_*, (\alpha_* \rtimes \beta_*)'), (Imv_*, (\alpha_* \rtimes \beta_*)''), v|Keru_*\big)
\end{array}$$

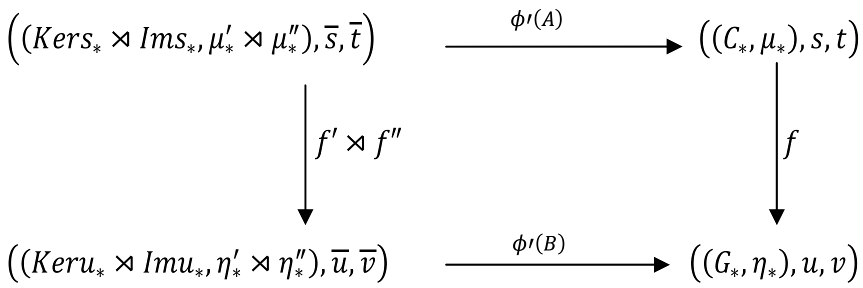
Now, we need only to show that $\Psi\phi = I_{R \circ T}$ and $\phi\Psi = I_{I_{CModComp}}$.
Let $A = \big((C_*, \mu_*), (G_*, \eta_*), \partial = \{\partial_n\}\big) \in Ob\ CModComp$, therefore
$\Psi(A)\phi(A) = \big(\pi_{C_*}^{-1}, \pi_{G_*}^{-1}\big)\big(\pi_{C_*}, \pi_{G_*}\big) = \big(I_{Kers_*}, I_{Ims_*}\big) = I_{(Kers_*, Ims_*, t|Kers_*)} = I_{R \circ T}(A)$.
Therefore $\Psi\phi = I_{R \circ T}$. Similarly, since $\pi_{C_n}\pi_{C_n}^{-1} = I_{C_n}$ and $\pi_{G_n}\pi_{G_n}^{-1} = I_{G_n}$ therefore $\phi\Psi = I_{I_{CmodComp}}$ and hence $R \circ T \approx I_{CmodComp}$.

Finally, we need to show that $T \circ R \approx I_{Cat^1\text{-}comp}$. To do this, define a function $\phi': T \circ R \longrightarrow I_{Cat^1\text{-}comp}$ as follows;

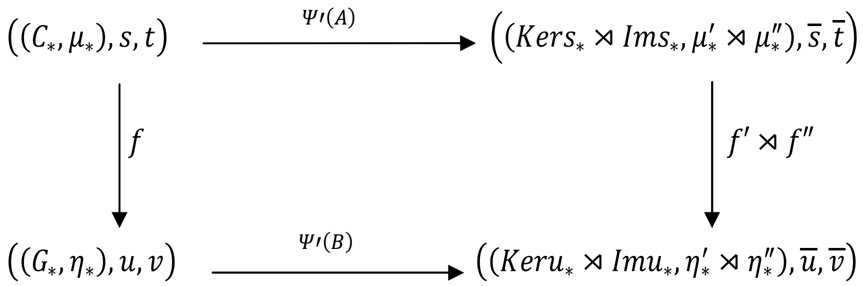
let $A = \big((C_*, \mu_*), s = \{s_n\}, t = \{t_n\}\big) \in ObCat^1\text{-}Comp$
$\phi'(A): \big((Kers_* \rtimes Ims_*, \mu_*' \rtimes \mu_*''), \bar{s} = \{\bar{s}_n\}, \bar{t} = \{\bar{t}_n\}\big) \to \big((C_*, \mu_*), s = \{s_n\}, t = \{t_n\}\big)$
such that $\phi'(A) = \xi_{C_*}$, where $\xi_{C_*} = \{\xi_{C_n}\}$ is a chain map defined by $\xi_{C_n}(a_n, b_n) = a_n b_n$ for all $(a_n, b_n) \in Kers_n \rtimes Ims_n$ and all $n \in Z$. Note that ξ_{C_*} is indeed a morphism of cat^1-complexes, for $s\xi_{C_*} = \xi_{C_*}\bar{s}$ and $t\xi_{C_*} = \xi_{C_*}\bar{t}$.

For any $f = \{f_n\}: A = \big((C_*, \mu_*), s = \{s_n\}, t = \{t_n\}\big) \to B = \big((G_*, \eta_*), u = \{u_n\}, v = \{v_n\}\big)$,
$$T \circ R(f) = f' \rtimes f'': \Big(\big(Kers_* \rtimes Ims_*, \acute{\mu}_* \rtimes \acute{\acute{\mu}}_*\big), \bar{s} = \{\bar{s}_n\}, \bar{t} = \{\bar{t}_n\}\Big) \longrightarrow \Big(\big(Keru_* \rtimes Imu_*, \acute{\eta}_* \rtimes \acute{\acute{\eta}}_*\big), \bar{u} = \{\bar{u}_n\}, \bar{v} = \{\bar{v}_n\}\Big).$$

We shall show that ϕ' is natural transformation. It is enough to show the commutativity of the following diagram:



Note that $f\,\phi'(A) = f\xi_{C_*}$ and $\phi'(B)(f' \rtimes f'') = \xi_{G_*}(f' \rtimes f'')$. Clearly, $f_n\xi_{C_n} = \xi_{G_n}(f'_n \rtimes f''_n)$ for all $n \in Z$. Thus ϕ' is natural transformation. Similarly, define a function $\Psi': I_{Cat^1-Comp} \longrightarrow T\circ R$ as follows;
Let $A = ((C_*, \mu_*), s = \{s_n\}, t = \{t_n\}) \in ObCat^1\text{-}Comp$,
$\Psi'(A){:}((C_*, \mu_*), s{=}\{s_n\}, t{=}\{t_n\}) \to ((Kers_* \rtimes Ims_*, \mu'_* \rtimes \mu''_*), \bar{s}{=}\{\bar{s}_n\}, \bar{t}{=}\{\bar{t}_n\})$ such that $\Psi'(A){=}\xi_{C_*}^{-1}$, where $\xi_{C_*}^{-1}$ is indeed a morphism of Cat^1-complexes, for $\xi_{C_*}^{-1}{=}\{\xi_{C_n}^{-1}\}$ is a chain map defined by $\xi_{C_n}^{-1}(c_n){=}(c_n s_n(c_n^{-1}), s_n(c_n))$ for all $c_n \in C_n$, all $n \in Z$, $\xi_{C_*}^{-1}s = \bar{s}\xi_{C_*}^{-1}$ and $\xi_{C_*}^{-1}t = \bar{t}\xi_{C_*}^{-1}$.
For any $f{=}\{f_n\}{:}\ A{=}((C_*, \mu_*), s{=}\{s_n\}, t{=}\{t_n\}) \to B{=}((G_*, \eta_*), u{=}\{u_n\}, v{=}\{v_n\})$, to show that Ψ' is natural transformation, it is enough to show the commutativity of the following diagram:



Note that $(f' \rtimes f'')\Psi'(A) = (f' \rtimes f'')\xi_{C_*}^{-1}$ and $\Psi'(B)f = \xi_{C_*}^{-1}f$.
Since $c_n s_n(c_n^{-1}) \in Kers_n, s_n(c_n) \in Ims_n$, $f'_n = f_n|Kers_n$, $f''_n = f_n|Ims_n$ and f is a morphism of cat^1-complexes, therefore $(f'_n \rtimes f''_n)\xi_{C_n}^{-1} = \xi_{C_n}^{-1}f_n$ for all $n \in Z$.
Thus Ψ' is a natural transformation. Furthermore, as $\xi_{C_n}^{-1}\xi_{C_n} = I_{Kers_n \rtimes Ims_n}$ we have $\Psi'\phi' = I_{T\circ R}$ and as $\xi_{C_n}\xi_{C_n}^{-1} = I_{C_n}$, we have $\phi'\Psi' = I_{I_{Cat^1\text{-}Comp}}$
Thus $T\circ R \approx I_{Cat^1-Comp}$ and $R\circ T \approx I_{CModComp}$. Hence $CModComp$ and $Cat^1\text{-}Comp$ are equivalent. ♦

THEOREM 3.3. T: $CModComp \to Cat^1\text{-}Comp$ is a left adjoint functor of $R{:}\ Cat^1\text{-}Comp \to CModComp$.

Proof. We shall show that there is a natural isomorphism $\Phi{:}\ Mor_{Cat^1\text{-}Comp}(T-,-) \to Mor_{CModComp}(-, R-)$, where

$Mor_{Cat^1\text{-}Comp}(T-,-)$, $Mor_{CModComp}(-,R-)$: $CModComp^{op}\times Cat^1\text{-}Comp \to S$ are bifunctors, where S denotes the category of sets and $CModComp^{op}$ denotes the opposite (dual) category of $CModComp$. These bifunctors are defined respectively by the following compositions;

$$CModComp^{op}\times Cat^1\text{-}Comp \xrightarrow{T^{op}\times I_{Cat^1\text{-}Comp}} Cat^1\text{-}Comp^{op}\times Cat^1\text{-}Comp \xrightarrow{E_{Cat^1\text{-}Comp}} S, \text{ and}$$

$$CModComp^{op}\times Cat^1\text{-}Comp \xrightarrow{I_{CModComp^{op}}\times R} CModComp^{op}\times CModComp \xrightarrow{E_{CModComp}} S.$$

Define a function $\Phi: Mor_{Cat^1\text{-}Comp}(T-,-)\to Mor_{CModComp}(-,R-)$ as follows;

For all $A=\Big(\big((C_*,\mu_*),(G_*,\eta_*),\partial\big)^{op},\big((H_*,\alpha_*),s,t\big)\Big)\in Ob\,CModComp^{op}\times Cat^1\text{-}Comp$,

$\Phi(A): Mor_{Cat^1\text{-}Comp}\Big(\big((C_*\rtimes G_*,\mu_*\rtimes\eta_*),\bar{s}=\{\bar{s}_n\},\bar{t}=\{\bar{t}_n\}\big),\big((H_*,\alpha_*),s=\{s_n\},t=\{t_n\}\big)\Big) \longrightarrow$

$Mor_{CModComp}\Big(\big((C_*,\mu_*),(G_*,\eta_*),\partial=\{\partial_n\}\big),\big((Kers_*,\alpha'_*),(Ims_*,\alpha''_*),t|Kers_*\big)\Big)$, is defined by $\Phi(A)(l)=(\bar{l},l^*)$, for all $l=\{l_n\}\in Mor_{Cat^1\text{-}Comp}\Big(\big((C_*\rtimes G_*,\mu_*\rtimes\eta_*),\bar{s},\bar{t}\big),\big((H_*,\alpha_*),s,t\big)\Big)$, where $\bar{l}=\{\bar{l}_n\}:(C_*,\mu_*)\to(Kers_*,\alpha'_*)$ and $l^*=\{l^*_n\}:(G_*,\eta_*)\to(Ims_*,\alpha''_*)$ are chain maps defined by $\bar{l}_n(c_n)=l_n(c_n,1_{G_n})$ and $l^*_n(g_n)=l_n(1_{C_n},g_n)$ for all $c_n\in C_n, g_n\in G_n$ and for each $n\in Z$. Note that $(\bar{l},l^*):\big((C_*,\mu_*),(G_*,\eta_*),\partial\big)\to\big((Kers_*,\alpha'_*),(Ims_*,\alpha''_*),t|Kers_*\big)$ is indeed a morphism of crossed modules of complexes since $\bar{l}$ and l^* are chain maps, $(t|Kers_*)\bar{l}_n=l^*_n\partial_n$ and $\bar{l}_n({}^{g_n}c_n)={}^{l^*_n(g_n)}\bar{l}_n(c_n)$ for all $c_n\in C_n, g_n\in G_n$ and all $n\in Z$. We turn now to show that Φ is a natural transformation, let $((f,k)^{op},\&)\in Mor_{CModComp^{op}\times Cat^1\text{-}Comp}(A,B)$, where $A=\Big(\big((C_*,\mu_*),(G_*,\eta_*),\partial\big)^{op},\big((H_*,\alpha_*),s,t\big)\Big)$ and $B=\Big(\big((D_*,\tau_*),(Y_*,\delta_*),\lambda\big)^{op},\big((X_*,\beta_*),u,v\big)\Big)$ It is enough to show the commutativity of the following diagram:

$$\begin{array}{ccccc}
A & & Mor_{Cat^1\text{-}Comp}\Big(\big((C_*\rtimes G_*,\mu_*\rtimes\eta_*),\bar{s},\bar{t}\big),U\Big) & \xrightarrow{\Phi(A)} & Mor_{CModComp}\Big(J,\big((Kers_*,\alpha'_*),(Ims_*,\alpha''_*),t|Kers_*\big)\Big)\\
\downarrow{\scriptstyle ((f,k)^{op},\&)} & & \downarrow{\scriptstyle E_{Cat^1\text{-}Comp}((f\rtimes k)^{op},\&)} & & \downarrow{\scriptstyle E_{CModComp}((f,k)^{op},(\&',\&''))}\\
B & & Mor_{Cat^1\text{-}Comp}\Big(\big((D_*\rtimes Y_*,\tau_*\rtimes\delta_*),\bar{u},\bar{v}\big),W\Big) & \xrightarrow{\Phi(B)} & Mor_{CModComp}\Big(Q,\big((Keru_*,\beta'_*),(Imu_*,\beta''_*),v|Keru_*\big)\Big)
\end{array}$$

where $U=\big((H_*,\alpha_*),s,t\big)$, $J=\big((C_*,\mu_*),(G_*,\eta_*),\partial\big)$, $W=\big((X_*,\beta_*),u,v\big)$, $Q=\big((D_*,\tau_*),(Y_*,\delta_*),\lambda\big)$.

Let $l=\{l_n\}\in Mor_{Cat^1\text{-}Comp}\Big(\big((C_*\rtimes G_*,\mu_*\rtimes\eta_*),\bar{s},\bar{t}\big),\big((H_*,\alpha_*),s,t\big)\Big)$. Therefore

$E_{CModComp}\big((f,k)^{op},(\&',\&'')\big)\Phi(A)(l)=E_{CModComp}\big((f,k)^{op},(\&',\&'')\big)(\bar{l},l^*)$

$=(\&',\&'')(\bar{l},l^*)(f,k)=(\&'\,\bar{l}\,f,\&''\,l^*\,k)$. On the other hand,

$\Phi(B)E_{Cat^1\text{-}Comp}((f\rtimes k)^{op},\&)\,(l)=\Phi(B)\big(\&\,l\,(f\rtimes k)\big)=\Big(\overline{\&\,l\,(f\rtimes k)},\big(\&\,l\,(f\rtimes k)\big)^*\Big).$

According to the definition of $\bar{l}$ and l^*, we have $\overline{\mathscr{k}\, l\, (f \rtimes k)} = \mathscr{k}'\, \bar{l}\, f$ and $\left(\mathscr{k}\, l\, (f \rtimes k)\right)^* = \mathscr{k}''\, l^*\, k$, respectively. Also, define a function $\Psi{:}Mor_{CModComp}(-,R-) \rightarrow Mor_{Cat^1\text{-}Comp}\,(T-,-)$ as follows;

for all $C=\left(\left((C_*,\mu_*),(G_*,\eta_*),\partial\right)^{op},\left((H_*,\alpha_*),s,t\right)\right) \in ObCModComp^{op}\times Cat^1\text{-}Comp$,

$$\Psi(C){:}Mor_{CModComp}\left(\left((C_*,\mu_*),(G_*,\eta_*),\partial=\{\partial_n\}\right),\left((Kers_*,\alpha'_*),(Ims_*,\alpha''_*),t|Kers_*\right)\right) \longrightarrow$$
$$Mor_{Cat^1\text{-}Comp}\left(\left((C_*\rtimes G_*,\mu_*\rtimes\eta_*),\bar{s}=\{\bar{s}_n\},\bar{t}=\{\bar{t}_n\}\right),\left((H_*,\alpha_*),s=\{s_n\},t=\{t_n\}\right)\right),$$

is defined by $\Psi(C)(\rho,\theta) = (\rho\rtimes\theta)^{\#}$, for all

$(\rho=\{\rho_n\},\theta=\{\theta_n\})\in Mor_{CModComp}\left(\left((C_*,\mu_*),(G_*,\eta_*),\partial\right),\left((Kers_*,\alpha'_*),(Ims_*,\alpha''_*),t|Kers_*\right)\right)$

where $(\rho\rtimes\theta)^{\#}{:}\left((C_*\rtimes G_*,\mu_*\rtimes\eta_*),\bar{s}=\{\bar{s}_n\},\bar{t}=\{\bar{t}_n\}\right)\rightarrow\left((H_*,\alpha_*),s=\{s_n\},t=\{t_n\}\right)$ is defined by $(\rho_n\rtimes\theta_n)^{\#}(c_n,g_n)=\rho_n(c_n)\theta_n(g_n)$ for all $(c_n,g_n)\in C_n\rtimes G_n$ and all $n\in Z$. Clearly $(\rho_n\rtimes\theta_n)^{\#}$ is a homomorphism and $\alpha_n(\rho_n\rtimes\theta_n)^{\#}=(\rho_{n-1}\rtimes\theta_{n-1})^{\#}(\mu_n\rtimes\eta_n)$ which implies that $(\rho\rtimes\theta)^{\#}$ is a chain map. Note that $(\rho\rtimes\theta)^{\#}$ is also a morphism of cat^1-complexes. We turn now to show that Ψ is a natural transformation. Let $((f,k)^{op},\mathscr{k})\in Mor_{CModComp^{op}\times Cat^1\text{-}Comp}(C,D)$, where $C=\left(\left((C_*,\mu_*),(G_*,\eta_*),\partial\right)^{op},\left((H_*,\alpha_*),s,t\right)\right)$ and $D=\left(\left((D_*,\tau_*),(Y_*,\delta_*),\lambda\right)^{op},\left((X_*,\beta_*),u,v\right)\right)$. It is enough to show the commutativity of the following diagram:

$$\begin{array}{cccc}
C & Mor_{CModComp}\left(J,\left((Kers_*,\alpha'_*),(Ims_*,\alpha''_*),t|Kers_*\right)\right) & \xrightarrow{\Psi(C)} & Mor_{Cat^1\text{-}Comp}\left(\left((C_*\rtimes G_*,\mu_*\rtimes\eta_*),\bar{s},\bar{t}\right),U\right) \\
\downarrow ((f,k)^{op},\mathscr{k}) & \downarrow E_{CModComp}\left((f,k)^{op},(\mathscr{k}',\mathscr{k}'')\right) & & \downarrow E_{Cat^1\text{-}Comp}\left((f\rtimes k)^{op},\mathscr{k}\right) \\
D & Mor_{CModComp}\left(Q,\left((Keru_*,\beta'_*),(Imu_*,\beta''_*),v|Keru_*\right)\right) & \xrightarrow{\Psi(D)} & Mor_{Cat^1\text{-}Comp}\left(\left((D_*\rtimes Y_*,\tau_*\rtimes\delta_*),\bar{u},\bar{v}\right),W\right)
\end{array}$$

Let $(\rho,\theta)\in Mor_{CModComp}\left(\left((C_*,\mu_*),(G_*,\eta_*),\partial\right),\left((Kers_*,\alpha'_*),(Ims_*,\alpha''_*),t|Kers_*\right)\right)$Therefore

$$E_{Cat^1\text{-}Comp}((f\rtimes k)^{op},\mathscr{k})\Psi(C)(\rho,\theta)= E_{Cat^1\text{-}Comp}((f\rtimes k)^{op},\mathscr{k})(\rho\rtimes\theta)^{\#}$$
$$=\mathscr{k}\,(\rho\rtimes\theta)^{\#}\,(f\rtimes k).$$ On the other hand,

$$\Psi(D)E_{CModComp}\left((f,k)^{op},(\mathscr{k}',\mathscr{k}'')\right)(\rho,\theta) = \Psi(D)\left((\mathscr{k}',\mathscr{k}'')(\rho,\theta)(f,k)\right)$$
$$= \Psi(D)(\mathscr{k}'\rho\, f, \mathscr{k}''\,\theta\, k)$$
$$= (\mathscr{k}'\rho\, f \rtimes \mathscr{k}''\,\theta\, k)^{\#}.$$

Let $(d_n,y_n)\in D_n\rtimes Y_n$. As $\rho_n\, f_n(d_n)\in Kers_n$, $\theta_n k_n(y_n)\in Ims_n$, $\mathscr{k}'_n=\mathscr{k}_n|Kers_n$, $\mathscr{k}''_n=\mathscr{k}_n|Ims_n$, and from the definition of $(\rho_n\rtimes\theta_n)^{\#}$, we have $(\mathscr{k}'_n\rho_n\, f_n \rtimes \mathscr{k}''_n\,\theta_n\, k_n)^{\#}(d_n,y_n)=\mathscr{k}_n(\rho_n\rtimes\theta_n)^{\#}(f_n\rtimes k_n)(d_n,y_n)$. Therefore $(\mathscr{k}'\rho\, f \rtimes \mathscr{k}''\,\theta\, k)^{\#} = \mathscr{k}\,(\rho\rtimes\theta)^{\#}\,(f\rtimes k)$.

Finally, we shall prove that $\phi\Psi=I_{Mor_{CModComp}(-,R-)}$ and $\Psi\phi=I_{Mor_{Cat^1\text{-}Comp}(T-,-)}$.

Let $A=\left(\left((C_*,\mu_*),(G_*,\eta_*),\partial\right)^{op},\left((H_*,\alpha_*),s,t\right)\right) \in ObCModComp^{op}\times Cat^1\text{-}Comp$

We need to show the commutativity of the following diagram:

$$\begin{array}{ccc} Mor_{Cat^1\text{-}Comp}\left(\left((C_*\rtimes G_*,\mu_*\rtimes\eta_*),\bar{s},\bar{t}\right),U\right) & \xrightarrow{\Phi(A)} & Mor_{CModComp}\left(J,\left((Kers_*,\alpha'_*),(Ims_*,\alpha''_*),t|Kers_*\right)\right) \\ & {\searrow}^{I_{Mor_{Cat^1\text{-}Comp}(T-,-)}(A)} & \downarrow \Psi(A) \\ & & Mor_{Cat^1\text{-}Comp}\left(\left((C_*\rtimes G_*,\mu_*\rtimes\eta_*),\bar{s},\bar{t}\right),U\right) \end{array}$$

Let $l \in Mor_{Cat^1\text{-}Comp}\left(\left((C_* \rtimes G_*, \mu_* \rtimes \eta_*), \bar{s}, \bar{t}\right), \left((H_*, \alpha_*), s, t\right)\right)$. Therefore $\Psi(A)\phi(A)(l) = \Psi(A)(\bar{l}, l^*) = (\bar{l} \rtimes l^*)^{\#}$, where $(\bar{l}_n \rtimes l_n^*)^{\#} = l_n$ on $C_n \rtimes G_n$ for all $n \in Z$. Thus $(\Psi(A)\phi(A))(l) = \left(I_{Mor_{Cat^1\text{-}Comp}(T-,-)}(A)\right)(l)$. Now, let $A=\left(\left((C_*,\mu_*),(G_*,\eta_*),\partial\right)^{op},\left((H_*,\alpha_*),s,t\right)\right) \in ObCModComp^{op}\times Cat^1\text{-}Comp$. We need to show the commutativity of the following diagram:

$$\begin{array}{ccc} Mor_{CModComp}\left(J,\left((Kers_*,\alpha'_*),(Ims_*,\alpha''_*),t|Kers_*\right)\right) & \xrightarrow{\Psi(A)} & Mor_{Cat^1\text{-}Comp}\left(\left((C_*\rtimes G_*,\mu_*\rtimes\eta_*),\bar{s},\bar{t}\right),U\right) \\ & {\searrow}^{I_{Mor_{CModComp}(-,R-)}(A)} & \downarrow \Phi(A) \\ & & Mor_{CModComp}\left(J,\left((Kers_*,\alpha'_*),(Ims_*,\alpha''_*),t|Kers_*\right)\right) \end{array}$$

For any $(\rho,\theta) \in Mor_{CModComp}\left(\left((C_*,\mu_*),(G_*,\eta_*),\partial\right),\left((Kers_*,\alpha'_*),(Ims_*,\alpha''_*),t|Kers_*\right)\right)$, we have $\phi(A)\Psi(A)(\rho,\theta) = \phi(A)((\rho \rtimes \theta)^{\#}) = \left(\overline{(\rho \rtimes \theta)^{\#}}, ((\rho \rtimes \theta)^{\#})^*\right)$.

In fact $\overline{(\rho_n \rtimes \theta_n)^{\#}} = \rho_n$ on C_n and $((\rho_n \rtimes \theta_n)^{\#})^* = \theta_n$ on G_n for all $n \in Z$.

Therefore $\phi(A)\Psi(A)(\rho,\theta) = \left(I_{Mor_{CModComp}(-,R-)}(A)\right)(\rho,\theta)$.

Thus $\Phi: Mor_{Cat^1\text{-}Comp}(T-,-) \to Mor_{CModComp}(-,R-)$ is a natural isomorphism, and hence T: $CModComp \to Cat^1\text{-}Comp$ is a left adjoint functor of $R: Cat^1\text{-}Comp \to CModComp$. ♦

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