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Global Analysis of an Endemic Model with Acute and Chronic Stages[1]

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Abstract

In this paper, according to consequent system of a endemic model, a model with Acute and Chronic Stages is proposed. By making use of differential equation and characteristic of hepatitis C, we obtain that, When $R_0 > 1$ the endemic equilibrium of system is globally stable.

Keywords: Epidemic; Nonlinear incidence; Global analysis

1. Introduction

In this paper, the stability of the equilibrium of a chronic stage on the disease transmission and behavior in an exponentially growing or decaying population is the focus of this paper. The framework is brought into the case of hepatitis C, a disease typically characterized by a long chronic stage. As is well known to us, Hepatitis C, formerly referred to as 'non-A, non-B' hepatitis, is an important infection of the liver which was first considered as a separate disease in 1975. In practice, the vast majority of patients with acute hepatitis C develop a chronic infection which is characterized by detection of HCV RNA for a period of at least six months after a newly acquired infection. The most common symptoms of acute hepatitis C are fatigue and jaundice. However, the majority of cases, including those with chronic disease, are asymptomatic. This makes the diagnosis of hepatitis C very difficult and can be explained clearly why the HCV epidemic is often called 'the silent epidemic' [1]. No vaccine is available for hepatitis C. The high mutability of the hepatitis C

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genome [2] composes its development. There is no evidence that the successful treatment of HCV gives any kind of partial or temporary immunity. Hence the models developed fall within the class of models that treated or recovered individuals move back to the susceptible class.

In fact, the only two works known to the authors are [3]. A model structured by age-since-infection has also been considered in relation to HIV in [4, 5]. Reade et al. discussed an ODE model for infections with acute and chronic Stages with feline calicivirus [3]. Their work is mostly numerical and focuses on the impact of vaccination on the acute and chronic phases. A model without exposed class is constructed in [6], which obtained the proportional stability of the equilibriums. In this paper, it is supposed that, after the primary infection, a host stays in a latent period before becoming infectious. A four-dimension model with acute and chronic stages is discussed. The epidemic is transmitted through people's direct contacts. We suppose that the disease has an exposed period and then the patients enter into the acute and finally they went through the chronic stage. The patients have no immunity after recovering and become susceptible again. We part the population in researched area into four classes: S-susceptible; E-exposed; I-infected with acute hepatitis C; V-infected with chronic hepatitis C. The total number in$t$time is$N(t) = S(t)+I(t)+V(t)+E(t)$.

2. Basic assumptions and the Mathematical model

Next, we construct the following model:

$$\begin{cases} \dot{S}(t) = bN - \frac{\beta I}{1+\alpha_1 I}\frac{S}{N} - \gamma VS - dS + \alpha V, \\ \dot{E}(t) = \frac{\beta I}{1+\alpha_1 I}\frac{S}{N} + \gamma V\frac{S}{N} - dE - \varepsilon E, \\ \dot{I}(t) = \varepsilon E - (d+k)I, \\ \dot{V}(t) = kI - (d+\alpha)V, \\ S(0) = S_0, E(0) = E_0, I(0) = I_0, V(0) = V_0, \end{cases} \tag{2.1}$$

By adding the equations of system (2.1)we obtain $\dot{N}(t) = (b-d)N$ We set $r = b - d$,then $\dot{N}(t) = rN$,hence $N = N_0 e^r t$, terefore r gives the growth rate of the population, if $r > 0$,that is $b > d$, the population exponentially grows, if $r < 0$,that is $b < d$, the population exponentially decreases. The case $r = 0$ or $b = d$ implies that the population is stationary. Setting $N = 1$,then the system (2.1)becomes the following equivalent system:

$$\begin{cases} \dot{S}(t) = b(1-S) - \frac{\beta IS}{1+\alpha_1 I} - \gamma VS + \alpha V, \\ \dot{E}(t) = \frac{\beta IS}{1+\alpha_1 I} + \gamma VS - (b+\varepsilon)E, \\ \dot{I}(t) = \varepsilon E - (b+k)I, \\ \dot{V}(t) = kI - (b+\alpha)V, \\ S(0) = S_0, E(0) = E_0, I(0) = I_0, V(0) = V_0, \end{cases} \tag{2.2}$$

Letting $E = 1 - S - I - V$ substitute E in the third equation of (2.2)and removing the second equation,we obtain

$$\begin{cases} \dot{S}(t) = b(1-S) - \frac{\beta IS}{1+\alpha_1 I} - \gamma VS + \alpha V, \\ \dot{I}(t) = \varepsilon(1-S-I-V) - (b+k)I, \\ \dot{V}(t) = kI - (b+\alpha)V, \\ S(0) = S_0, E(0) = E_0, I(0) = I_0, V(0) = V_0, \end{cases} \tag{2.3}$$

Setting Γ={$(S, I, V) \in R^3 \mid S > 0$,$I > 0$,$V > 0$,$S + I + V \leq 1$}, obviously$\Gamma$ is a invariable set of (2.3).

3. Stability of the endemic and limit cycle

In this section, we study the stability and bifurcation of the endemic .
Let

$$R_0 = \varepsilon\frac{\beta(b+\alpha)+k\gamma}{(b+\alpha)(\varepsilon+b)(k+b)} = \frac{\beta\varepsilon}{(k+b)(\varepsilon+b)} + \frac{k\epsilon\gamma}{(b+\alpha)(\varepsilon+b)(k+b)}.$$

When $R_0 > 1$, the model (2.3) has a unique endemic equilibrium . Evaluating the of the model (2.3) at E^* gives

$$J = \begin{bmatrix} -b - \frac{\beta I^*}{1+\alpha_1 I^*} - \gamma V^* & \frac{-\beta S^*}{(1+\alpha_1 I^*)^2} & -\gamma S^* + \alpha \\ -\varepsilon & -\varepsilon - b - k & -\varepsilon \\ 0 & k & -(b+\alpha) \end{bmatrix}.$$

Then the characteristic equation about E^* is given by $\lambda^3 + E\lambda^2 + F\lambda + G = 0$, where

$$E = 3b + \frac{\beta I^*}{1+\alpha_1 I^*} + \gamma V^* + \alpha + \varepsilon + k,$$

$$F = (b+\frac{\beta I^*}{1+\alpha_1 I^*}+\gamma V^*)(b+k+\varepsilon)-\frac{\beta S^*\varepsilon}{(1+\alpha_1 I^*)^2}+(b+\alpha)(2b+k+\varepsilon+\frac{\beta I^*}{1+\alpha_1 I^*}+\gamma V^*)+k\varepsilon,$$

$$G = (b+\alpha)[(b+\frac{\beta I^*}{1+\alpha_1 I^*}+\gamma V^*)(b+k+\varepsilon)-\frac{\beta S^*\varepsilon}{(1+\alpha_1 I^*)^2}]+k[\varepsilon(b+\alpha)+\varepsilon(\frac{\beta I^*}{1+\alpha_1 I^*}+\gamma V^*)-\varepsilon\gamma S^*].$$

Evidently $E > 0$, establishing the sign of G in the following $G > [(b+\alpha)(b+k+\varepsilon)+k\varepsilon](\frac{\beta I^*}{1+\alpha_1 I^*}+\gamma V^*) > 0$. If $S^* < \frac{1}{R_0}$ the last inequality is tenable. In the following, we calculate the sign of $EF - G$,

$$EF-G = (b+\alpha)^2(2b+k+\varepsilon+\frac{\beta I^*}{1+\alpha_1 I^*}+\gamma V^*)+k\varepsilon(2b+k+\varepsilon)+(2b+k+\varepsilon+\frac{\beta I^*}{1+\alpha_1 I^*}+\gamma V^*)$$

$$\times[(b+\frac{\beta I^*}{1+\alpha_1 I^*}+\gamma V^*)(b+k+\varepsilon)-\frac{\beta S^*\varepsilon}{(1+\alpha_1 I^*)^2}+(b+\alpha)(2b+k+\varepsilon+\frac{\beta I^*}{1+\alpha_1 I^*}+\gamma V^*)].$$

Theorem 3.1. Suppose $R_0 > 1$, then the disease endemic equilibrium E^* of (2.3) is a stable node or focus when $EF - G > 0, G > 0$. E^* is an unstable node or focus when $EF - G < 0$, or $G < 0$ and has at least one closed orbit in Ω ; E^* is a center of the linear system of (2.3) when $EF - G = 0$.
Letting $V = 1-S-E-I$ substitute the V of the first and the second equations in (2.2), then the system (2.2) becomes

$$\begin{cases} \dot{S}(t) = b(1-S) - \frac{\beta IS}{1+\alpha_1 I} - \gamma S(1-E-I) + \alpha(1-S-E-I) + \gamma S^2, \\ \dot{E}(t) = \frac{\beta IS}{1+\alpha_1 I} + \gamma S(1-E-I) - \gamma S^2 - (b+\varepsilon)E, \\ \dot{I}(t) = \varepsilon E - (b+k)I, \\ \S(0) = S_0, E(0) = E_0, I(0) = I_0, V(0) = V_0, \end{cases} \tag{3.1}$$

Setting Ω =$\{(S,E,I) \in R_1^3 \mid S+E+I < 1\}$. Let $x \to f(x) \in R^n$ be a C^1 function for x in an open set $D \subset R^n$. Consider the differential equation

$$\dot{x} = f(x) \tag{3.2}$$

Denote by $x(t, x_0)$ the solution to (3.2) such that $x(0, x_0) = x_0$. We make the following two assumptions: (H_1)$eq.$ (2.3) has a unique equilibrium $\bar{x}$ in D. (H_2) There exists a compact absorbing set $K \subset D$.
Lemma 3.2. Under the assumptions (H_1) and (H_2), find conditions on (3.2) such that the local stability of $\bar{x}$ implies its global stability in D.
Theorem 3.3. When $R_0 > 1$, if $\gamma = 0, \alpha = 0$, then the endemic equilibrium P^* of system (2.3) is globally stable in Γ^0.
proof: The global stability of the endemic equilibrium of system (2.3) in P^* is equivalent to that of the endemic equilibrium $\bar{P}(S^*, E^*, I^*)$ of system (3.1) in Ω . Evidently, system (3.1) has unique endemic equilibrium $\bar{P}$ in Ω , hence it satisfies the assumption (H_1). Because $R_0 > 1$, the instability of and the boundness of the solutions of system (3.1) ensure the system (3.1) has a compact set in Ω, so it also satisfies the assumption (H_2). When $\gamma = 0, \alpha = 0$, the Jocobian matrix of system (3.1)is

$$J = \begin{bmatrix} -b - \frac{\beta I^*}{1+\alpha_1 I^*} & 0 & \frac{-\beta S^*}{(1+\alpha_1 I^*)^2} \\ \frac{\beta I^*}{1+\alpha_1 I^*} & -\varepsilon - b & \frac{\beta S^*}{(1+\alpha_1 I^*)^2} \\ 0 & \varepsilon & -(b+k) \end{bmatrix}.$$

And its second additive compound matrix is

$$J^{[2]} = \begin{bmatrix} -2b - \frac{\beta I^*}{1+\alpha_1 I^*} - \varepsilon & \frac{\beta S^*}{(1+\alpha_1 I^*)^2} & \frac{\beta S^*}{(1+\alpha_1 I^*)^2} \\ \varepsilon & -2b - \frac{\beta I^*}{1+\alpha_1 I^*} - k & 0 \\ 0 & \frac{\beta I^*}{1+\alpha_1 I^*} & -2b-k-\varepsilon \end{bmatrix}.$$

Set the function $P(X) = P(S, E, I) = diag\{1, \frac{E}{I}, \frac{E}{I}\}$, then $P_f P^{-1} = diag\{0, \frac{\dot{E}}{E} - \frac{\dot{I}}{I}, \frac{\dot{E}}{E} - \frac{\dot{I}}{I}\}$, And the matrix $B = P_f P^{-1} + PJ^{[2]}P^{-1}$ can be written in block form

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where

$$B_{11} = -2b - \frac{\beta I^*}{1+\alpha_1 I^*} - \varepsilon, B_{12} = (\frac{\beta S}{(1+\alpha_1 I)^2}\frac{I}{E}, \frac{\beta S}{(1+\alpha_1 I)^2}\frac{I}{E}),$$

$$B_{21} = \begin{pmatrix} \varepsilon\frac{E}{I} \\ 0 \end{pmatrix}, B_{22} = \begin{pmatrix} \frac{\dot{E}}{E} - \frac{\dot{I}}{I} - 2b - \frac{\beta I^*}{1+\alpha_1 I^*} - k & 0 \\ \frac{\beta I^*}{1+\alpha_1 I^*} & \frac{\dot{E}}{E} - \frac{\dot{I}}{I} - 2b - k - \varepsilon \end{pmatrix}$$

Let (u, v, w) denote the vectors in $R^3 \cong R^{\binom{3}{2}}$, we select a norm in R^3 as $|(u, v, w)| = max\{|u|, |v + w|\}$. And let u denote the measure with respect to this norm. Using the method of estimating u in [7], we have

$$u(B) \leq sup\{g_1, g_2\}, \tag{3.3}$$

where

$$g_1 = u_1(B_{11}) + |B_{12}|, g_2 = |B_{21}| + u_1(B_{22}),$$

Here B_{12}, B_{21} are matrix norms with respect to the l_1 vector norm, and u_1 denotes the Lozinskii measure with respect to the l_1 norm. More specifically,

$$u_1(B_{11}) = -2b - \frac{\beta I^*}{1+\alpha_1 I^*} - \varepsilon, |B_{12}| = \frac{\beta S}{(1+\alpha_1 I)^2}\frac{I}{E},$$

$$|B_{21}| = \varepsilon\frac{E}{I}, u_1(B_{22}) = \frac{\dot{E}}{E} - \frac{\dot{I}}{I} - 2b - k.$$

Therefore $g_1 = \frac{\beta S}{(1+\alpha_1 I)^2} - 2b - \beta - \frac{\beta I^*}{1+\alpha_1 I^*}, g_2 = \frac{\dot{E}}{E} - \frac{\dot{I}}{I} - 2b - k + \varepsilon\frac{E}{I}$. Hence

$$g_1 \leq \frac{\dot{E}}{E} - b - \frac{\beta I^*}{1+\alpha_1 I^*} \leq \frac{\dot{E}}{E} - b, g_2 = \frac{\dot{E}}{E} - b.$$

By (3.3) we can obtain $u(B) \leq \frac{\dot{E}}{E} - b$. Along each solution $x(t, x_0), (x_0 \in K)$, where K is the compact absorbing set, we thus have $\frac{1}{t}\int_0^t u(B)ds \leq \frac{1}{t}\log\frac{E(t)}{E_0} - b$, when $t \to \infty, \bar{g_2} \leq \frac{-2}{b} < 0$.

According to the theorem of Li and Muldowney, if $\gamma = 0, \alpha = 0$, the endemic equilibrium $\bar{P}$ of system (2.3) is globally stable in Ω. The proof is completed.

Theorem 3.4. Suppose $\hat{R_0} < R_0 < 1$ and $b + \alpha > A_1$. Then the endemic equilibrium E_* of (2.3) is a saddle, E^* is a stable node or focus when $EF-G>0, G>0$; E^* is an unstable node or focus when $EF-G<0$, or $G<0$, and has at least one closed orbit in Ω ; E^* is a center of the linear system of (2.3) when $EF-G=0$.

proof: the Jocobian of E_* is given by

$$M_1=\begin{bmatrix} -b-\frac{\beta I_*}{1+\alpha_1 I_*}-\gamma V_* & \frac{-\beta S_*}{(1+\alpha_1 I_*)^2} & -\gamma S_*+\alpha \\ -\varepsilon & -\varepsilon-b-k & -\varepsilon \\ 0 & k & -(b+\alpha) \end{bmatrix}.$$

For $S_*=1-\frac{(k+b)(\varepsilon+\alpha+b)+\varepsilon\alpha}{\varepsilon(b+\alpha)}I_*$, We have

$$\det(M_1)=-(b+\frac{\beta I_*}{1+\alpha_1 I_*}+\gamma V_*)[k\varepsilon+(b+\alpha)(b+k+\varepsilon)]+k\gamma-\frac{k\gamma[k\varepsilon+(b+\alpha)(b+k+\varepsilon)]}{\varepsilon(b+\alpha)}I_*$$

$$-k\alpha+\frac{(b+\alpha)\beta\varepsilon}{(1+\alpha_1 I_*)^2}-\frac{\beta I_*}{(1+\alpha_1 I_*)^2}[k\varepsilon+(b+\alpha)(b+k+\varepsilon)].$$

for $V_*=\frac{k}{b+\alpha}I_*$, $I_*=\frac{-b_1-\sqrt{\Delta}}{2b_0}$, substitute I_* , then after some algebra we can see that $\det(M_1)<0$ and the equilibrium $E_*(S_*,I_*,V_*)$ is a saddle point. The proof is completed. Next we analyze the stability of the second positive equilibrium $E^*(S^*,I^*,V^*)$. The Jocobian matrix at $E^*(S^*,I^*,V^*)$ is

$$M_2=\begin{bmatrix} -b-\frac{\beta I^*}{1+\alpha_1 I^*}-\gamma V^* & \frac{-\beta S^*}{(1+\alpha_1 I^*)^2} & -\gamma S^*+\alpha \\ -\varepsilon & -\varepsilon-b-k & -\varepsilon \\ 0 & k & -(b+\alpha) \end{bmatrix},$$

By a similar argument as above, we obtain that $\det(M_2)>0$. Thus, E^* is a node, or a focus, when $EF-G>0$; E^* is an unstable node or focus when $EF-G<0$; E^* is a center of the linear system when $EF-G=0$. The proof is completed.

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