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Union Curves of a Hypersurface of a Weyl Space

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Abstract

In this paper, we have defined the union curves of a hypersurface $W_n(g_{ij}, T_k)$ of a Weyl space $W_n(g_{ab}, T_c)$ with respect to a congruence.

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INTRODUCTION

A manifold with a conformal metric g_{ij} and a symmetric connection ∇_k satisfying the compatibility condition

$$\nabla_k g_{ij} - 2T_k g_{ij} = 0 \tag{1.1}$$

is called a Weyl space that will be denoted by $W_h(g_{ij}, T_k)$. The vector field T_k is named the complementary vector field. Under renormalization of the metric tensor g_{ij} in the form

$$\overset{\nu}{g}_{ij} = \lambda^2 g_{ij} \tag{1.2}$$

the complementary vector field T_k is transformed by the law

$$\overset{\nu}{T}_k = T_k + \partial_k \ln \lambda \tag{1.3}$$

where λ is a scalar function [1].

If, under transformation (1.2), the quantity A is changed according to the rule

$$\overset{\nu}{A} = \lambda^p A \tag{1.4}$$

then A is called satellite of g_{ij} with weight $\{p\}$.

The prolonged derivative and prolonged covariant derivative of A are, respectively, defined by ([2],[3])

$$\dot{\partial}_k A = \partial_k A - pT_k A \tag{1.5}$$

and

$$\dot{\nabla}_k A = \nabla_k A - pT_k A. \tag{1.6}$$

Let $W_n(g_{ij}, T_k)$ be a hypersurface of the Weyl space $W_{n+1}(g_{ab}, T_c)$ and let x^a $(a = 1.2. \dots , n + 1)$ and u^i $(i = 1.2. \dots , n)$ be, respectively, the coordinates of $W_{n+1}(g_{ab}, T_c)$ and $W_n(g_{ij}, T_k)$. The metrics of $W_n(g_{ij}, T_k)$ and $W_{n+1}(g_{ab}, T_c)$ are connected by the relations

$$g_{ij} = g_{ab} x^a_i x^b_j \ (j = 1, 2, \dots , n\,;\ b = 1, 2, \dots , n + 1) \tag{1.7}$$

where x^a_i is the covariant derivative of x^a with respect to u^i.

The prolonged covariant derivative of A with respect to u^k and x^c are, respectively, $\dot{\nabla}_k A$ and $\dot{\nabla}_c A$ and related by the conditions

$$\dot{\nabla}_k A = x^c_k \dot{\nabla}_c A \ (k = 1, 2, \dots , n\,;\ c = 1, 2, \dots , n + 1). \tag{1.8}$$

Let the normal vector field n^a of $W_n(g_{ij}, T_k)$ be normalized by the condition $g_{ab} n^a n^b = 1$.

Since the weight of x^a_i is $\{0\}$, the prolonged covariant derivative of x^a_i, relative to u^k, is given by

$$\dot{\nabla}_k x^a_i = \nabla_k x^a_i = w_{ik} n^a \tag{1.9}$$

where w_{ik} are the coefficients of the second fundamental form of $W_n(g_{ij,}T_k)$.

On the other hand, it is easy to see that the prolonged covariant derivative of n^a is given by

$$\dot{\nabla}_k n^a = -w_{kl} g^{il} x_i^a. \tag{1.10}$$

Let $\underset{r}{v}^i$ $(i, r = 1, 2, \dots, n)$ be the contravariant compenents of the vector field v_r in $W_n(g_{ij}, T_k)$. Suppose that vector fields $\underset{r}{v}$ $(r = 1, 2, \dots, n)$ are normalized by the conditions $g_{ij}\underset{r}{v}^i\underset{r}{v}^j = 1$.

The prolonged covariant derivative of the vector field given by [4]

$$\dot{\nabla}_k \underset{r}{v}^i = \overset{p}{\underset{r}{T}}_k \underset{p}{v}^i \; (r, p = 1, 2, \dots, n) \tag{1.11}$$

Let $\underset{r}{v}^a$ and $\underset{r}{v}^i$ be, respectively, the contravariant components of the vector field v_r relative to $W_{n+1}(g_{ab}, T_c)$ and $W_n(g_{ij}, T_k)$, we have [5]

$$\underset{r}{v}^a = x_i^a \underset{r}{v}^i, \; (a = 1, 2, \dots, n+1\,;\, i = 1, 2, \dots, n) \tag{1.12}$$

If $\underset{rr}{\kappa}$ is the normal curvature of the hypersurface $W_n(g_{ij}, T_k)$ in the direction of $\underset{r}{v}$, we have

$$\underset{rr}{\kappa} = w_{ij}\underset{r}{v}^i\underset{r}{v}^j. \tag{1.13}$$

Since the weight of w_{ij} is $\{1\}$ and that of $\underset{r}{v}^i$ is $\{-1\}$, $\underset{rr}{\kappa}$ is a satellite of g_{ij} with weight of $\{-1\}$.

The quantities

$$\overset{r}{\underset{p}{\wr}} = \overset{r}{\underset{p}{T}}_k \underset{r}{v}^k \qquad (r, p = 1, 2, \dots, n) \tag{1.14}$$

are called the geodesic curvatures of the lines of the net $\left(\underset{1}{v}, \underset{2}{v}, \dots, \underset{n}{v}\right)$ [4].

The vector fields

$$\underset{p}{c}^i = \overset{r}{\underset{p}{\wr}} \underset{r}{v}^i \qquad (i, r, p = 1, 2, \dots, n) \tag{1.15}$$

are called the geodesic vector fields of the net $\left(\underset{1}{v}, \underset{2}{v}, \dots, \underset{n}{v}\right)$ relative to $W_n(g_{ij}, T_k)$ [4].

If the components of the geodesic vector fields relative to $W_{n+1}(g_{ab}, T_c)$ are denoted by $\underset{r}{\overline{c}}{}^{a}$, then we have [5]

$$\underset{r}{v}{}^{c} \overset{\bullet}{\nabla}_c \underset{r}{v}{}^{a} = \underset{r}{\overline{c}}{}^{a} = \left(w_{ik} \underset{r}{v}{}^{i} \underset{r}{v}{}^{k}\right) n^a + \underset{r}{c}{}^{i} x_i^a. \tag{1.16}$$

Since the net $\left(\underset{1}{v}, \underset{2}{v}, \dots, \underset{n}{v}\right)$ is ortogonal, we have by [4]

$$\underset{r}{\overset{r}{T}}_k = 0\,,\ \underset{r}{\overset{p}{T}}_k + \underset{p}{\overset{r}{T}}_k = 0 \quad (r \neq p). \tag{1.17}$$

2. Totally Geodesic Surface in W_{n+1}

Let $W_n(g_{ij}, T_k)$ be a hypersurface of the Weyl space $W_{n+1}(g_{ab}, T_c)$. Let C be a curve in W_n.

Definition 2.1 Totally geodesic surface in W_{n+1} is determined by the tangent vector field of the curve C relative to W_{n+1} and by the derivative of the tangent vector field of the curve C relative to W_{n+1} along the curve C.

Let us consider a congruence of the curves in W_{n+1} such that one curve of the congruence passes through each point of W_n and let us denote it by v. Let v^a be the contravariant components of v in the x's and let v^a be normalized by the condition $g_{ab}v^a v^b = 1$. The vector field v with components v^a, in general, not normal to W_n and can be specified by

$$v^a = t^i x_i^a + r n^a \tag{2.1}$$

where t^i and r are parameters [6].

Since $g_{ab}v^a v^b = 1$, with the help (1.11) and (1.13)

$$t_i t^i = 1 - r^2 \tag{2.2}$$

is valid.

Let y be a vector field in W_n. If it's contravariant components in the x's and the covariant components in the u's are denoted by y^a and y^j, respectively, then there is the following relation between y^a and y^i

$$y^a = x^a_i y^i. \tag{2.3}$$

If we take the absolute derivative of y^a along the curve C relative to W_{n+1} and if h^a are contravariant components in the x's of the derived vector field relative to W_{n+1} and h^j are the contravariant components in the u's of the derived vector field relative to W_n, the following relation is valid:

$$\begin{aligned} h^a &= \underset{s}{v}^b \dot{\nabla}_b y^a = \underset{s}{v}^k \dot{\nabla}_k \left(x^a_j y^j\right) \\ &= \underset{s}{v}^k \left(\dot{\nabla}_k x^a_i\right) y^i + x^a_i \underset{s}{v}^k \left(\dot{\nabla}_k y^i\right) \\ &= w_{ki} \underset{s}{v}^k y^i n^a + x^a_i h^i \end{aligned} \tag{2.4}$$

where $\underset{s}{v}$ is the tangent vector field of the curve C in W_n, it is normalized by the condition $g_{ij} \underset{s}{v}^i \underset{s}{v}^j = 1$ and w_{ki} are the coefficients of the second fundamental form of $W_n(g_{ij}, T_k)$.

If the geodesic in W_{n+1} in the direction of the congruence with direction v^a is to be a geodesic of the totally geodesic surface, then v can be written as a linear combination of y^a and h^a :

$$v^a = t^i x^a_i + r n^a = \alpha y^a + \beta h^a. \tag{2.5}$$

Besides, since y^i is equal to $\underset{s}{v}^i$, the expression (2.5) transforms to

$$v^a = t^i x^a_i + r n^a = \alpha y^a + \beta h^a = \alpha x^a_i \underset{s}{v}^i + \beta h^a \tag{2.6}$$

where $h^a = \underset{s}{\overline{c}}{}^a = \underset{rr}{\kappa} h^a + x^a_i \underset{s}{c}^i$, $h^i = \underset{s}{c}^i = \underset{s}{\overset{p}{\eta}} \underset{p}{v}^i$ $(p = 1, 2, \ldots, n)$ [5]. Here $\underset{rr}{\kappa}$ is the normal curvature of the curve C in W_n in the direction of

$\underset{s}{v}$, $\underset{s}{\overline{c}}^{\,a}$ and $\underset{s}{c}^{\,i}$ are the geodesic curvature vector fields of the curve C relative to W_{n+1} and W_n, respectively.

Therefore, (2.6) can be written as

$$v^a = t^j x_i^a + rn^a = \alpha x_i^a \underset{s}{v}^i + \beta \left[w_{ki} \underset{s}{v}^k \underset{s}{v}^i n^a + x_i^a \underset{s}{c}^k \right]. \tag{2.7}$$

Let us calculate the coefficients α and β :

If the equations (2.7) are multiplied by $g_{ab} x_j^b$ and the summation is taken on a and b,

$$g_{ij} t^i = \alpha g_{ij} \underset{s}{v}^i + \beta g_{ij} \underset{s}{c}^i \tag{2.8}$$

is obtained.

If we multiply (2.8) by $\underset{s}{v}^j$ and we take the summation on i and j, we get the first coefficient as

$$g_{ij} t^i \underset{s}{v}^j = \alpha \tag{2.9}$$

where $\overset{s}{\underset{s}{T}} = 0$ [4], $g_{ij} \underset{p}{v}^i \underset{s}{v}^j = 0$, $p = (1, 2, \ldots, s-1, s+1, \ldots, n)$.

If we multiply (2.7) by $g_{ab} n^b$ and we take the summation on a and b, we find the second coefficient as

$$r = \beta w_{ki} \underset{s}{v}^k \underset{s}{v}^i = \beta \underset{ss}{\kappa}$$

or

$$\beta = \frac{r}{\underset{ss}{\kappa}} \tag{2.10}$$

where $g_{ab} n^a n^b = 1$, $g_{ab} x_i^a n^b = 0$.

If we put into place the values of α and β in the equations (2.8), we get

$$g_{ij} t^i = g_{mn} t^m t^n g_{ij} \underset{s}{v}^i + \frac{r}{\underset{ss}{\kappa}} g_{ij} \underset{s}{c}^i. \tag{2.11}$$

If the equations (2.11) are multiplied by g^{jk} and the summation is taken on i and j, we obtain

$$t^k = g_{mn} t^m \underset{s}{v}^n \underset{s}{v}^k + \frac{r}{\underset{ss}{\kappa}} \underset{s}{c}^k \tag{2.12}$$

or

$$\underset{ss}{\kappa} \frac{t^k}{r} = \underset{ss}{\kappa} g_{mn} \frac{t^m}{r} \underset{s}{v}^n \underset{s}{v}^k + \underset{s}{c}^k \tag{2.13}$$

where $g_{ij} g^{jk} = \delta_i^k$.

If we take $\frac{t^k}{r} = l^k$, (2.13) transforms to

$$\underset{ss}{\kappa}\, l^k = \underset{ss}{\kappa}\, g_{mn} l^m \underset{s}{v}^n \underset{s}{v}^k + \underset{s}{c}^k \tag{2.14}$$

or

$$\underset{s}{c}^k - \underset{ss}{\kappa} \left(l^k - g_{mn} l^m \underset{s}{v}^n \underset{s}{v}^k \right) = 0 \quad (k = 1, 2, \dots, n). \tag{2.15}$$

3. Union Curves in W_n

For a congruence specified by the parameters l^k, the solutions of the n equations (2.15) determine the union curves in W_n relative to that congruence.

Let us denote the left hand side of the equations (2.15) by η^k, which we shall call the contravariant components of the union curvature vector field in W_n:

$$\eta^k = \underset{s}{c}^k - \underset{ss}{\kappa} \left(l^k - g_{mn} l^m \underset{s}{v}^n \underset{s}{v}^k \right) = 0 \quad (k = 1, 2, \dots, n). \tag{3.1}$$

Definition 3.1 The union curve of W_n relative to a congruence as defined as a curve whose union curvature vector field is a null vector field.

By means of definition 3.1 and (3.1), the following corollaries are obtained:

Corollary 3.1 If the union curve C is an asymptotic curve in W_n, in which case $\underset{ss}{\kappa}$ is equal to zero, then C is a geodesic in W_n.

Corollary 3.2 If the union curve C is a geodesic in W_n, then it is either an asymptotic curve or the vector field with components $l^k - g_{mn} l^m \underset{s}{v}^n \underset{s}{v}^k$ is a null vector field.

Now, let us calculate the magnitude of the union curvature vector field:

Let us denote it by κ_u. Then:

$$
\begin{aligned}
\kappa_u^2 &= g_{ij}\eta^i\eta^j \\
&= g_{ij}\left[\underset{s}{c}^i - \underset{ss}{\kappa}\left(l^i - g_{mn} l^m \underset{s}{v}^n \underset{s}{v}^i\right)\right]\left[\underset{s}{c}^j - \underset{ss}{\kappa}\left(l^j - g_{pq} l^p \underset{s}{v}^q \underset{s}{v}^j\right)\right] \\
&= g_{ij}\underset{s}{c}^i\underset{s}{c}^j - \underset{ss}{\kappa} g_{ij}\underset{s}{c}^i l^j + \underset{ss}{\kappa} g_{ij}\underset{s}{c}^i\underset{s}{v}^j g_{pq} l^p \underset{s}{v}^q - \underset{ss}{\kappa} g_{ij} l^i \underset{s}{c}^j \\
&+ \underset{ss}{\kappa} g_{ij}\underset{s}{v}^i\underset{s}{c}^j g_{mn} l^m \underset{s}{v}^n + \underset{ss}{\kappa}^2 g_{ij} l^i l^j - \underset{ss}{\kappa}^2 g_{ij} l^i \underset{s}{v}^j g_{pq} l^q \underset{s}{v}^q \\
&- \underset{ss}{\kappa}^2 g_{ij}\underset{s}{v}^i l^j g_{mn} l^m \underset{s}{v}^n + \underset{ss}{\kappa}^2 g_{ij}\underset{s}{v}^i\underset{s}{v}^j g_{mn} l^m \underset{s}{v}^n g_{pq} l^p \underset{s}{v}^q \\
&= \kappa_g^2 - 2\underset{ss}{\kappa} g_{ij}\underset{s}{c}^i l^j + \underset{ss}{\kappa}^2 g_{ij} l^i l^j - \underset{ss}{\kappa}^2 g_{ij} l^i \underset{s}{v}^j g_{pq} l^p \underset{s}{v}^q \\
&- \underset{ss}{\kappa}^2 g_{ij}\underset{s}{v}^i l^j g_{mn} l^m \underset{s}{v}^n + \underset{ss}{\kappa}^2 g_{mn} l^m \underset{s}{v}^n g_{pq} l^p \underset{s}{v}^q
\end{aligned}
$$

where κ_u is the union curvature of the curve C, $\kappa_g = \overset{p}{\underset{s}{\wr}}$ is the geodesic curvature of the curve C which has tangent vector field $\underset{s}{v}$ relative to W_n, $g_{ij}\underset{s}{c}^i\underset{s}{v}^j = g_{ij}\overset{p}{\underset{s}{\wr}}\underset{p}{v}^i\underset{s}{v}^j = \overset{p}{\underset{s}{T}}_k \underset{s}{v}^k g_{ij}\underset{p}{v}^i\underset{s}{v}^j = 0$ since $\overset{s}{\underset{s}{T}} = 0$ [4] and $g_{ij}\underset{p}{v}^i\underset{s}{v}^j = 0$ $p = (1, 2, \ldots, s-1, s+1, \ldots, n)$. Furthermore, we know that $g_{ij}\underset{s}{v}^i\underset{s}{v}^j = 1$.

If we continue the operations, we get

$$
\begin{aligned}
\kappa_u^2 &= \kappa_g^2 - 2\underset{ss}{\kappa} g_{ij}\underset{s}{c}^i l^j + \underset{ss}{\kappa}^2 g_{ij} l^i l^j - \underset{ss}{\kappa}^2 g_{ij} l^i \underset{s}{v}^j g_{pq} l^p \underset{s}{v}^q \\
&= \kappa_g^2 - 2\underset{ss}{\kappa} \overset{p}{\underset{s}{\wr}}\, g_{ij}\underset{p}{v}^i l^j + \underset{ss}{\kappa}^2 g_{ij} l^i l^j - \underset{ss}{\kappa} g_{ij} l^i \underset{s}{v}^j g_{pq} l^p \underset{s}{v}^q \\
&= \kappa_g^2 - 2\underset{ss}{\kappa} \overset{p}{\underset{s}{\wr}} \cos\left(\frac{\pi}{2} - \alpha\right) \tan\phi + \underset{ss}{\kappa}^2 \tan^2\phi - \underset{ss}{\kappa} \cos^2\alpha \tan^2\phi \\
&= \kappa_g^2 - 2\underset{ss}{\kappa}\, \kappa_g \sin\alpha\, \tan\phi + \underset{ss}{\kappa}^2 \tan^2\phi\, \left(1 - \cos^2\alpha\right) \\
&= \kappa_g^2 - 2\underset{ss}{\kappa}\, \kappa_g \sin\alpha\, \tan\phi + \underset{ss}{\kappa}^2 \tan^2\phi\, \sin^2\alpha
\end{aligned}
$$

$$\kappa_u^2 = \left(\kappa_g - \underset{ss}{\kappa} \tan\phi\, \sin\alpha\right)^2 \tag{3.2}$$

or

$$\kappa_u = \kappa_g - \underset{ss}{\kappa} \tan\phi\, \sin\alpha \tag{3.3}$$

where α is the angle between $\underset{s}{v}$ and l^k, ϕ is the angle between the vector fields v^a and n^a, therefore
$\cos\phi = r$, $g_{ij}e^i e^j = \tan^2\phi$, $\cos\alpha\tan\phi = g_{ij}\underset{s}{v}^i l^j$ and
$g_{ij}\underset{p}{v}^i l^j = \cos\left(\frac{\pi}{2} - \alpha\right) \tan\phi \quad p = (1, 2, \dots, s-1, s+1, \dots, n)$.

The expression (3.3) gives the relation between the union curvature, the geodesic curvature and the normal curvature of the C in W_n. From that relation:

Corollary 3.3 If $\phi = 0$, the union curve is a geodesic.

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