International Mathematical Forum, Vol. 7, 2012, no. 1, 37 - 45

Otsuki Connections of Submanifolds of Projective Spaces

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Abstract. Using Otsuki's connection $(P, {}'\Gamma, {}''\Gamma)$ we obtain new equations for normalized surfaces, which, in a special case, reduce to the equations given by Norden. Using obtained relations $\nabla_k g_{ij}$, in case of $\det |g_{ij}| \neq 0$ we prove that Otsuki's coefficients of transfer to surface X_m are uniformly specified in external normalization. In the case of internal normalization, where all points of surface X_m belong to absolute hypersurface we derive again new reduced equations of normalized surfaces, which reduce in the special case to the equations given by Norden. In the case $\det |g_{ij}| \neq 0$ using obtained relation $\nabla_k g_{ij} = p_i^t l_k g_{ij} + P_j^t l_k g_{ti}$ we prove that the Otsuki's connection coefficients of surface whose all points belong to the absolutes are uniformly defined.

Mathematics Subject Classification: 53B25, 53B05

Introduction

The connections ${}'\Gamma, {}''\Gamma$ naturally appear in differential geometry of smooth manifolds. The connections ${}'\Gamma, {}''\Gamma$ determined by a relative normalization (x, X, y) are torsion free Ricci systematic connections. We refer [SSV] and [PAS] for more details.

A triple $({}'\Gamma, g, {}''\Gamma)$ of two symmetric connections ${}'\Gamma$ and ${}''\Gamma$ and a semi-Riemannian metric g is called a conjugate triple if the compatibility condition is satisfied:

$$\frac{\partial g_{ij}}{\partial x^k} - {}'\Gamma^a_{ik} g_{aj} - {}''\Gamma^a_{jk} g_{ia} = 0$$

For conjugate triples in relative hypersurface theory, see Section 4. Conjugate triples also appear on hypersurface in space forms with Weingarten operators of maximal rank; see e.g. [SSV].

Otsuki considered general regular connections on n-dimensional C^∞ manifolds as a pair of two connections $({}'\Gamma, {}''\Gamma)$, where ${}'\Gamma$ is the contravariant part and ${}''\Gamma$ is the covariant part.

Norden [No1], [No2] studied the parallel displacement according to two connections $'\Gamma, ''\Gamma$ of two directions which are conjugated with respect to a systematic nondegenerate pseudotensor g.

Prvanović [Prv] constructed a family of pairs of linear connections which are conjugated in the sense of Otsuki satisfying at the same time Norden's condition of conjugation. She also considered conformal transformations and calculated a certain conformally invariant tensor. We refer also [Prv1], [Prv2], [Prv3], [Pu2] for other results related to the geometry of Otsuki spaces.

All of these mentioned results motivate a further study of Otsuki spaces which reduce in a special case to the Norden spaces. In Section 1 we introduce notations and notions, which we use throughout the paper. We state also elementary relations between these notions. In Section 2 we study an Otsuki connection on a submanifold X_m of a projective space $\mathcal{P}_n$ according to external normalization. Internal normalization of X_m is studied in Section 3. For a normalization $x = \lambda x$ we have $y_i = \lambda y_i$ and $\check{l}_i = l_i + \partial_i \log \lambda$ of the obtained structure. Consequently, the second normal is not changed.

For a point $x \in N(X_m)$ we have the reper $x, y_i, \underset{s}{X}$ $(s = 1, \dots, n-m;$ $i = 1, \dots, m)$ of $n+1$ points. We have this reper to derive the fundamental equations of a normalized submanifold $N(X_m)$;

$$\partial_i x = y_i + l_i x \tag{0.1}$$

$$\nabla_j y_i = P_i^t l_j y_t + p_{ji} x + \overset{s}{b}_{ij} \underset{s}{X}, \quad j, i = 1, 2 \dots, m, \tag{0.2}$$

$$\nabla_j \underset{s}{X} = p_i^t \underset{s}{m}{}_j^i y_t + \underset{s}{m}_j x + \underset{s}{n}{}_l^k \underset{k}{X}, \quad t, k = 1, 2, \dots, n-m. \tag{0.3}$$

For $P_i^t = \delta_i^t$ we have an affine mapping when equations (0.1), (0.2) and (0.3) reduce to equations given by Norden.

P_β^α is a mixed tensor such that $\det(p_\beta^\alpha) \neq 0$ which satisfies the relation

$$\frac{\partial p_\beta^\alpha}{\partial x^\gamma} + {}''\Gamma_{\mu\gamma}^\alpha p_\beta^\mu - p_\sigma^\alpha {}'\Gamma_{\beta\gamma}^\sigma = 0,$$

where $'\Gamma$ and $''\Gamma$ are two connections.

Let $\underset{x}{n}^\alpha$ be linearly independent vectors. Vectors $\underset{x}{n}^\alpha$ with $x_\alpha^i = \dfrac{\partial x^\alpha}{\partial n_i}$ form the base of tangential space of a manifold X_m in a point. The base $\overset{x}{n}_\alpha, x_\alpha^i$ of dual vector space is uniquely determined by condition

$$\underset{x}{n}^\alpha \overset{y}{n}_\alpha = \delta_x^y, \quad x_i^\alpha \overset{x}{n}_\alpha = 0, \quad x_i^\alpha x_\alpha^j = \delta_i^j, \quad x_\alpha^i \underset{x}{n}^\alpha = 0,$$

where $x_i^\alpha x_\beta^i + \underset{s}{n}^\alpha \overset{x}{n}_\beta = \delta_\beta^\alpha$.

1. Preliminaries

T. Otsuki [Ots] has introduced a general regular connection in n-dimensional C^∞ manifold M^n. This connection is defined by two different parts: contravariant part $'\Gamma$ and covariant part $''\Gamma$. In this case the covariant derivative of $(1,1)$ type tensor V is given by the relation

$$D_k V^i_j = \left(\frac{\partial V^a_b}{\partial x^k} + {'\Gamma}^q_{sk} V^s_b - {''\Gamma}^s_{kb} \right) P^i_a P^b_j \tag{1.1}$$

where the tensor P of type $(1,1)$ $(\det(P^i_j) \neq 0)$ is the fundamental tensor of Otsuki theory. $'\Gamma$ and $''\Gamma$ are affine connections satisfying the relation

$$\frac{\partial p^\alpha_\beta}{\partial x^\gamma} + {''\Gamma}^i_{ak} P^a_j - P^i_a {'\Gamma}^a_{jk} = 0, \tag{1.2}$$

which is equivalent with the condition $D_k Q^i_j$, where $Q = P^{-1}$, i.e.,

$$P^i_s Q^s_j = \delta^i_j. \tag{1.3}$$

On the other hand let $N(X_m)$ be a normalized submanifold X_m of a projective space $\mathcal{P}_n$. This means that in every point $x \in \mathcal{P}$ there exist two linear manifolds P_{n-m} and P_{m-1} such that

(a) $\mathcal{P}_{n-m}$ contains the point $x \in \mathcal{P}_n$, but with the tangent plane T_m in x has not other common points,

(b)$\mathcal{P}_{m-1}$ is a subset of T_m, but it does not contain x.

We call manifolds $\mathcal{P}_{n-m}$ and $\mathcal{P}_{m-1}$ normals P_I of the first type and $\mathcal{P}_{II}$ of the second type for $N(X_m)$ respectively.

We denote by $\underset{p}{X}$ $(p = 1, 2, \ldots, n-m)$ points of $\mathcal{P}_I$ and call them tops of the first type normal. The points $\underset{p}{X}$ and x compose a system of $n-m+1$ independent points and determine P_I.

Let tangent coordinate lines on X_m in a point x cut P_{II} in points $y_i = \partial_i x - l_i x$ which are independent and determine P_{II}. We call them support points.

Given P_{II} is equivalent to a given set l_i of m scalars. A choice of l_i is connected with a choice of curvilinear coordinate system and the normalization of the vector x which determines a point of X_m.

l_i is a tensor which we call a normalizator of $N(X_M)$.

In projective space $\mathcal{P}_n$ projective metric is introduced and defined by the polarity[1] with the fundamental tensor $a_{\alpha\beta}$. Thus, scalar product of vector of corresponding points is defined by $xy = a_{\alpha\beta} x^\alpha y^\beta$ and the Weierstrass normalization $x^2 = \sigma = \pm 1$. Normalization of a surface X_m given in $\mathcal{P}_n$ is polar with respect to given polarity if P_I and P_{II} corresponding to each point of conjugated polar.

Given polarity is absolute polarity of that normalization.

[1]Polarity is correspondence between points and hypersurface defined by relation $\zeta_\alpha = a_{\alpha\beta} x^\beta$, where $a_{\alpha\beta}$ is symmetric tensor.

P_I is orthogonal on tangent space T_m because P_{II} lies in T_m which is polar for P_I.

It the points of surface X_m do not belong to absolute hypersurface of second order Q_{n-1} which is defined by absolute polarity, surface normalization is external.

If all points of X_m belong to absolute hypersurface normalization is internal.

2. External normalization

Point of surface is defined by vector $x = x(u^1, u^2, \dots, u^m)$. The Weierstrass normalization is valid, consequently we have relation $x^2 = \sigma = \pm 1$.

All points P_{II} are conjugated with fixed point X through which passes P_I, than $x \cdot y_i = 0$. As $xy_i = x(\partial_i x - l_i x) = \frac{1}{2}\partial x^2 - l_i x^2 = \frac{1}{2}\partial_i \sigma - l_i \sigma = -l_i \sigma$, than $l_i \sigma = 0$ and $l_i = 0$.

From $l_i = 0$ follows $y_i y_j = \partial_i x \partial_j x = g_{ij}$ which proves that the points' scalar products of support are coordinates of tensor projective metrics.

Vectors $\underset{s}{X}$, $s = 1, 2, \dots, n-m$ are mutually independent and define the tops of normal of first order. We take them that the tops P_I lie on surface $\mathcal{P}_{n-2}$ at which normal P_I intersects polar of point x.

Now we have relations $\underset{s}{X} y_i = 0$ and $\underset{s}{X} x = 0$.

Vectors $\underset{s}{X}$ are fully defined by relation $\underset{s}{X}\,\underset{t}{X} = g_{st} = \text{const}$.

Let the reper whose tops coincide with tops of normal, is autopolar; then we have a relation

$$g_{ss} = \varepsilon_s = \pm 1; \quad g_{st} = 0 \ (s \neq t) \quad (s, t = 1, 2, \dots, n-m).$$

Then from previously mentioned follows

	x	y_i	$\underset{s}{X}$
x	ε	0	0
y_j	0	g_{ij}	0
$\underset{t}{X}$	0	0	g_{st}

We differentiate relation $xy_i = 0$ using the table to obtain

$$(\partial_j x) y_i + x(\nabla_j y_i) = 0,$$

$$(y_j + l_j x) y_i + x(P_i^t l_j y_t + p_{ji} x + \overset{s}{b}_{ij} \underset{s}{X}) = 0$$

$$y_i \cdot y_j + \varepsilon p_{ij} = 0 \iff g_{ij} = -\varepsilon p_{ji} \iff p_{ji} = -\varepsilon g_{ij}. \tag{2.1}$$

We differentiate relation $\underset{s}{X} x = 0$ to obtain

$$(\nabla_j \underset{s}{X}) + \underset{s}{X}(\partial_j x) = 0 \iff (P_i^t \underset{s}{m}{}_j^i y_t + \underset{s}{m}_j x + \underset{s}{n}{}_j^k \underset{k}{X}) x + \underset{s}{X}(y_j + l_j x) = 0.$$

Consequently we have

$$\underset{s}{m}_l \varepsilon = 0 \Rightarrow (\underset{s}{m}_j = 0). \tag{2.2}$$

We differentiate relation $\underset{s}{X}\,\underset{t}{X} = \text{const}$ to obtain

$$(\nabla_j \underset{s}{X})\,\underset{t}{X} + \underset{s}{X}(\nabla_j \underset{t}{X}) = 0 \Rightarrow (P_i^r \underset{s}{m}{}_j^i y_r + \underset{s}{n}{}_j^k \underset{k}{X})\,\underset{t}{X} + \underset{s}{X}(P_i^r \underset{t}{m}{}_j^i y_r + \underset{t}{n}{}_j^k \underset{k}{X}) = 0.$$

Thus we have

$$g_{tk}\,\underset{s}{n}{}_j^k + g_{sk}\,\underset{t}{n}{}_j^k = 0. \tag{2.3}$$

Finally we differentiate relation $\underset{s}{X} y_i = 0$ to obtain

$$y_i(P_i^t \underset{s}{m}{}_j^i y_t + \underset{s}{n}{}_j^k \underset{k}{X}) + (P_i^t l_j y_t + p_{ji}x + b_{ij}^k \underset{k}{X})\,\underset{s}{X} = 0.$$

Hence it follows $P_i^t \underset{s}{m}{}_j^i g_{ti} + b_{ij}^k g_{ks} = 0$. This relation implies

$$P_i^t \underset{s}{m}{}_{jt} + \underset{s}{b}{}_{ij} = 0. \tag{2.4}$$

The relations (2.1), (2.2), (2.3), and (2.4) imply the fundamental equations of normalized surface in external normalization

$$\partial_i x = y_i,$$
$$\nabla_j y_i = -\varepsilon g_{ij} x + \overset{s}{b}_{ij}\,\underset{s}{X}$$
$$\nabla_j \underset{s}{X} = P_i^t \underset{s}{m}{}_j^i y_t + \underset{s}{n}{}_j^k \underset{k}{X}.$$

Now we differentiate relation $g_{ij} = y_i y_j$ to have

$$\nabla_k g_{ij} = (-\varepsilon g_{ij} x) y_j + (b_{ik}^s \underset{s}{X}) y_j + y_i(-\varepsilon g_{jk} x) + y_i(b_{jk}^s \underset{s}{X}) = 0.$$

From relation $\nabla_k g_{ij} = 0$ follows that connection of normalized surface in external normalization is metrical. Its metric is induced by absolute polarity.

Assuming that $\det|g_{ij}| \neq 0$ surface X_m is called nonisotropic surface with respect to metric defined by absolute polarity.

Theorem 1. *If* $\det|g_{ij}| \neq 0$*, then coefficients of Otsuki's displacement on surface* X_m *normalized in external polar normalization are uniquely determined (if tensor* P_β^α *is corresponded to manifold* X_m*, then regular Otsuki's connection* $(P, {}'\Gamma, {}''\Gamma)$ *of ambient space induces regular Otsuki's connection* $(\bar{P}, {}'\bar{\Gamma}, {}''\bar{\Gamma})$ *of submanifold).*

When $\nabla_k g^{ij} = 0$ is on the surface, then we have an affine mapping.

Proof. From $\nabla_k g_{ij} = p_i^s p_j^l g_{sl|k}$ and $\nabla_k g_{ij} = 0$ it follows

$$P_i^s P_j^l (\partial_k g_{sl} - {}''\bar{\Gamma}_{sk}^h g_{hl} - {}''\bar{\Gamma}_{lk}^h g_{sh}) = 0 \text{ e.e.,}$$
$$\partial_k g_{sl} - {}''\bar{\Gamma}_{sk}^h g_{hl} - {}''\bar{\Gamma}_{lk}^h g_{sh} = 0 \tag{2.5}$$

k, s, l indexes permutation imply other two relations, which due to ${}''\bar{\Gamma}_{kl}^h = {}''\bar{\Gamma}_{lk}^h$ give

$$-2{}''\bar{\Gamma}_{ls}^h g_{hk} + \partial_l g_{ks} + \partial_s g_{lk} - \partial_k g_{sl} = 0.$$

Consequently

$$''\bar{\Gamma}^h_{ls}g_{hk} - \frac{1}{2}(\partial_l g_{ks} + \partial_s g_{lk} - \partial_k g_{sl})$$

i.e.,

$$''\bar{\Gamma}^h_{ls} = \frac{1}{2}(\partial_l g_{ks} + \partial_s g_{lk} - \partial_k g_{sl}).$$

The relations $\dfrac{\partial P^i_j}{\partial x^k} + {}''\bar{\Gamma}^i_{rk}P^r_j - {}''\bar{\Gamma}^s_{jk}P^i_s = 0$ determine the coefficients ${}'\bar{\Gamma}^h_{ls}$.

We know that $g^{ir}g_{rj} = \delta^i_j$. Differentiating this relation we get

$$(\nabla_k g^{ir})g_{rj} + g^{ir}(\nabla_k g_{rj}) = \nabla_k \delta^i_j$$

i.e.,

$$(\nabla_k g^{ir})g_{rj} = P^i_m P^s_j \delta^m_{s|k},$$

$$\nabla_k g^{ir} = g^{rj} P^i_m P^s_j (\partial_k \delta^m_s + {}'\bar{\Gamma}^m_{hk}\delta^h_s - {}''\bar{\Gamma}^h_{sk}\delta^m_k).$$

In an affine space from relation $\nabla_k g_{ij}$ follows $\nabla_k g^{ij} = 0$. If $\nabla_k g^{ij} = 0$ from relation (*) follows $\delta^i_{j|k} = 0$.

Since $\delta^i_{j||k} = {}'\bar{\Gamma}^i_{jk} - {}''\bar{\Gamma}^i_{jk}$ we have that ${}'\bar{\Gamma} = {}''\bar{\Gamma}$, i.e., Otsuki displacement reduces to an affine mapping. □

3. Internal polar normalization

Polar normalization is internal if point $x = x(u^1, u^2, \ldots, u^m)$ belongs to absolute. Introducing scalar product using tensor of absolute polarity we have relation $x^2 = 0$.

Second order normal P_{II} is polar normal of first order P_I which passes trough point x, and all points of normal P_{II} (here are the support points Y_i) satisfy the condition $xy_i = 0$.

From relation $x^2 = 0$ it follows $x\partial_i x = 0$. Now we have

$$y_i y_j = \partial_i x \partial_j x - l_i x \partial_j x - l_j x \partial_i x + l_j l_i x^2 = \partial_i x \partial_j x$$

i.e., $y_i y_j = g_{ij}$, where g_{ij} is fundamental tensor of projective metric.

Since points of surface belong to absolute, it implies that tensor g_{ij} is defined in the points of surface. Due to indeterminacy of normalization of vector x satisfying conditions $x^2 = 0$, this determinacy is up to scalar factor.

We chose the tops $\underset{s}{X}$ of first order normality $s = 1, 2, \ldots, n - m - 1$ in tangential surface, such that their vectors fulfill conditions $\underset{s}{X} y_i = 0$, $\underset{s}{X} x = 0$.

Final top X does not belong to tangential hyperplane, because its vector has to fulfill inequality $Xx \neq 0$.

We normalize this space to obtain $Xx = 1$ and we chose this top on absolute such that we have $X^2 = 0$.

We chose all tops $\underset{s}{X}$ in subspace $\mathcal{P}_{n-2}$ of intersection of tangent hyperplanes of absolute in points x and X. Then the following relations hold

$$y_i X = 0, \quad \underset{s}{X} \cdot X = 0; \quad \underset{s}{X} \cdot \underset{t}{X} = g_{st} = \text{const}, \quad (s,t = 1,2,\dots,n-m-1).$$

Now we have table where we emphasize the top X and summarize all previous mentioned facts.

	x	y_i	X	$\underset{s}{X}$
x	0	0	1	0
y_j	0	g_{ij}	0	0
X	1	0	0	0
$\underset{t}{X}$	0	0	0	g_{st}

Emphasizing the top X we obtain the equations for normalized surface

$$\partial_i x = y_i + l_i X$$
$$\nabla_j y_i = P_i^t l_j y_t + p_{ji} x + b_{ji} X + \overset{s}{b}_{ji} \underset{s}{X}$$
$$\nabla_j \underset{s}{X} = P_i^t \underset{s}{m}{}_j^i y_t + \underset{s}{m}{}_j x + \underset{s}{n}{}_j X + \underset{s}{n}{}_j^t \underset{t}{X}$$
$$\nabla_j X = P_i^t m_j^i y_t + m_j x + n_j X + n_j^t \underset{t}{X}.$$

Differentiating relation we obtain

(1) From $xy_i = 0$ it follows $b_{ji} = -g_{ji}$
(2) From $x \cdot X = 1$ it follows $n_j = -l_j$
(3) From $x\underset{s}{X} = 0$ it follows $\underset{s}{n}{}_j = 0$
(4) From $X \cdot y_i = 0$ it follows $P_{ji} = -P_i^t m_j^t m_j^i g_{ti}$
(5) From $\underset{s}{X} y_i = 0$ it follows $\underset{s}{b}{}_{ji} = -P_i^t \underset{s}{m}{}_j^i g_{ti}$
(6) From $X^2 = 0$ it follows $m_j = 0$
(7) From $X \cdot \underset{s}{X}$ it follows $\underset{s}{m}{}_j = -\underset{s}{n}{}_j$
(8) From $\underset{s}{X} \cdot \underset{t}{X} = g_{st}$ it follows $\underset{st}{n}{}_j + \underset{ts}{n}{}_j = 0$
(9) From $y_i y_j = g_{ij}$ it follows $P_i^t l_k g_{tj} + P_j^t l_k g_{ti} = \nabla_k g_{ij}$.

The last relation is used in order to obtain coefficients of displacement $''\bar{\Gamma}$.

Equations of normalized surface in internal polar normalization due to previous relations are the following

$$\partial_i x = y_i + l_i x$$
$$\nabla_j y_i = P_i^t l_j y_t + p_{ji} x - g_{ji} X + \overset{s}{b}_{ji} \underset{s}{X}$$
$$\nabla_j \underset{s}{X} = P_i^t \underset{s}{m}{}_j^i y_t - \underset{s}{n}{}_j x + \underset{s}{n}{}_j^t \underset{t}{X}$$
$$\nabla_j X = P_i^t m_j^i y_t - l_j X + \overset{t}{n}_j \underset{t}{X}$$

with additional conditions (4), (5), and (9).

Finally we prove theorem.

Theorem 2. *Let g_{ij} be the fundamental tensor of projective metric such that* $\det(g_{ij}) \neq 0$. *Then coefficients $''\bar{\Gamma}^t_{sr}$ of Otsuki displacements are uniquely determined and consequently the Otsuki displacement is also determined on a normalized surface whose all points belong to absolute.*

Proof. We use the relation (9) and $\nabla_K g_{ij} = P^r_i P^s_j g_{rs|k}$ to get

$$
\begin{aligned}
P^r_i P^s_j g_{rs|k} &= P^t_i l_k g_{tj} + p^t_j l_k g_{ti} \\
P^r_i P^s_j (\partial_k g_{rs} - {''}\bar{\Gamma}^t_{rk} g_{ts} - {''}\bar{\Gamma}^t_{sk} g_{rt}) &= P^t_i l_k g_{tj} + P^t_j l_k g_{ti} \\
Q^i_b P^r_i Q^j_n P^s_j (\partial_k g_{rs} - {''}\bar{\Gamma}^t_{rk} g_{ts} - {''}\bar{\Gamma}^t_{sk} g_{rt}) &= Q^i_b Q^j_n (P^t_i l_k g_{tj} + P^t_j l_k g_{ti}) \\
\delta^r_k \delta^n_s (g_k g_{rs} - {''}\bar{\Gamma}^t_{rk} g_{ts} - {''}\bar{\Gamma}^t_{sk} g_{rt}) &= l_k (Q^i_b Q^j_n P^t_i g_{tj} + Q^i_b Q^j_n P^t_j g_{ti}) \\
\partial_k g_{bn} - {''}\bar{\Gamma}^t_{bk} g_{tn} - {''}\bar{\Gamma}^t_{nk} g_{bt} &= l_k (\delta^t_b Q^j_n g_{tj} + Q^i_b \delta^t_n g_{ti}) \\
\partial_k g_{bn} - {''}\bar{\Gamma}_{bk} g_{tn} - {''}\bar{\Gamma}^t_{nk} g_{bt} &= l_k (Q^j_n g_{bj} + Q^i_b g_{ni}).
\end{aligned}
$$

Let $Q^j_n g_{bj} = Q_{nb}$ and $Q^i_b g_{ni} = Q_{bn}$; assuming $Q_{nb} = Q_{bn}$ we have $\partial_k g_{bn} - {''}\bar{\Gamma}^t_{bk} g_{tn} - {''}\bar{\Gamma}^t_{nk} g_{bt} = 2 l_k Q_{nb}$. Cyclical permutation of k, b, n indexes from previous relations, assuming ${''}\bar{\Gamma}^t_{bn} = {''}\bar{\Gamma}^t_{nb}$ we obtain new relation

$$
\partial_k g_{bn} - \partial_b g_{nk} - \partial_n g_{kb} + 2{''}\bar{\Gamma}^t_{nb} g_{tk} = 2(l_k Q_{nb} - l_n Q_{bk} - l_b Q_{kn})
$$

wherefrom it follows

$$
{''}\bar{\Gamma}^t_{nb} g_{tk} = \frac{1}{2}(\partial_b g_{nk} + \partial_n g_{kb} - \partial_k g_{bn}) + l_k Q_{nb} - l_n Q_{bk} - l_b Q_{kn},
$$

or finally

$$
{''}\bar{\Gamma}^t_{nb} = g^{tk}\Big[\frac{1}{2}(\partial_b g_{nk} + \partial_n g_{kb} - \partial_k g_{bn}) + l_k Q_{nb} - l_n Q_{bk} - l_b Q_{kn}\Big].
$$

${'}\bar{\Gamma}^t_{nb}$ is induced displacement through ${''}\bar{\Gamma}^t_{nb}$ as relations of the fundamental Otsuki displacement on a normalized surface are given by

$$
\frac{\partial P^i_j}{\partial x_k} + {''}\bar{\Gamma}^i_{rk} P^r_j - {''}\bar{\Gamma}^s_{jk} P^i_s = 0.
$$

□

Acknowledgments

I am indebted to Mileva Prvanović and Neda Bokan for their help in bringing this work to life.

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Received: July, 2011