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Analytic Solution of Two Dimensional Advection Diffusion Equation Arising In Cytosolic Calcium Concentration Distribution

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Abstract

Mathematical and computational modeling of calcium signalling in astrocytes has produced considerable insights into how the astrocytes contracts with other cells under the variation of biophysical and physiological parameters. The modeling of calcium signaling in astrocytes has become more sophisticated. The modeling effort has provided insight to understand the cell contraction. Objective of this work is to study the calcium profile in the form of advection diffusion equation. A mathematical model is developed in the form of advection diffusion equation for the calcium profile. The model incorporates the important physiological parameter like diffusion coefficient etc. Appropriate boundary conditions have been framed. Analytic solution is found using Laplace transform in the form of complementary error function. MATLAB 7.5 has been used to simulate the model and obtain the results.

Mathematics Subject Classification: 92C20,74H10

Keywords: calcium profile, advection diffusion, laplace transform, astrocytes

1 Introduction

The problem of neuroscience pose new challenges for mathematics and models of these problems are more interesting. One of the notable examples is of modeling calcium signaling in glial cell like astrocytes. Astrocytes are found the most diverse population of glial cells in nerves system. Twenty years ago, the traditional view of astrocytes as merely supportive cells providing only structural and metabolic support to neurons [8,16]. Recent studies of astrocytes have suggested that these cells have a more active and direct role in the dynamic regulation of cerebral microcirculation, synaptic transmission and neuronal activation [1, 2, 7, 11, 14]. Many biophysical and physiological process are taking place like flow of calcium through cell, calcium buffering, sodium-calcium exchanger (NCX) etc. Calcium [Ca^{2+}] plays pivotal role in cell signaling. It is used in signal transduction where an electrical signal is converted in the chemical signal. For instance, in pancreatic acinar cells, the frequency of calcium oscillations has been shown to determine the secretion rate of digestive enzymes and fluids [5]. Similar dynamic behaviour has been found in Astrocytes [9]. E. Samson, J. Marchand describe the multiionic transport model of cement-based materials exposed to aggressive environments. The concentration profile of each ionic species is taken into account in the form of diffusion, electrical coupling between the ions, chemical activity effects and advection caused by a capillary suction flow. Shuai Zeng et al. investigated the possible role of voltage-gated Ca^{2+} channels in spontaneous Ca^{2+} oscillations of astrocytes. T. Hofer et. al. have investigated the intercellular Ca^{2+} wave propagation through gap-junctional Ca^{2+} diffusion.

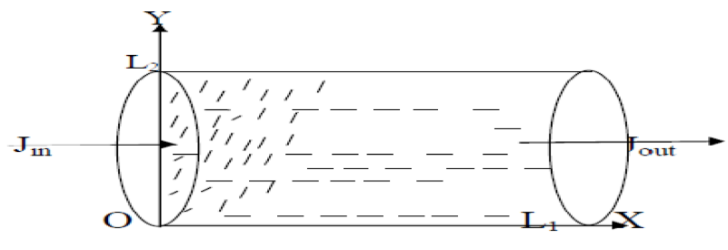

Figure 1: Conservation in one dimension

A number of research workers have attempted to carry out experimental and theoretical investigations on calcium distribution in astrocytes. Experiential investigations were carried out by Wang Z. et al to determine the effect of buffer on spatial and temporal characteristics of intercellular calcium signals in astrocytes. The theoretical analysis and its interpretation of the data has been based on the following conservation law in differential form [6, 10].

$$\frac{\partial C}{\partial t} - \frac{\partial J}{\partial x} = f(x, t, C) \tag{1}$$

Here C is the concentration of Ca^{2+}, J(x, t) is the rate at which C moves across the boundary at position x from left to right at time t, f (x, t, c) denote the net rate of increase of C (production-destruction) per unit volume at location x and time t. Here it is suppose that there is a uniform macroscopic flow of the Ca^{2+}, with speed v along the x-axis, which carries additional [Ca^{2+}]

along with it. When both diffusive flux and advective flux is incorporated, then the total flux will be

$$J(x,t) = vC(x,t) - D\frac{\partial C(x,t)}{\partial x} \tag{2}$$

Using this constitutive relation, (1) becomes a reaction-advection-diffusion equation,

$$\frac{\partial C}{\partial t} - \frac{\partial (vC(x,t) - D\frac{(\partial Cx,t)}{\partial x}\Big)}{\partial x} = f(x,t,C) \tag{3}$$

From the above literature survey, it is clear that almost all the one-dimensional and two-dimensional models have been developed incorporating diffusion of calcium. Actually when calcium $[Ca^{2+}]$ enters into the cytosol it diffuse latterly also. This has not been taken so far by research workers as evident from the literature survey. In view of above, Jha, Adlakha and Mehta have developed a model to study advection diffusion of calcium in astrocytes[3]. Now two dimensional advection diffusion model is developed to study the calcium profile. Analytic solution is found using Laplace transform.

2 Mathematical Formulation

The shape of Astrocytes is very irregular in geometrical figure for the shake of development of mathematical model we chose its some part as semi-finite shallow shape shown in figure 1. Where mouth of the channel is situated at the face and the semi-finite shallow has length l. When calcium inters into the cytosol from mouth of the channel the calcium concentration diffuses in x as well as y direction both. When x approaches to l_1 some length and y approaches to l_2, the calcium concentration diffused in x as well as y direction for any time , but here the diffusion will take place together with advection because calcium molecule will linearly transferred in x direction as well as y direction. Hence diffusion takes place when this molecule will transfer from one place to another place not in a linear form then advection also takes place.

Therefore two dimensional advection diffusion equation of calcium concentration can be given by

$$\frac{\partial \left[Ca^{2+}\right]}{\partial t} = D_{Ca_x}\frac{\partial^2 \left[Ca^{2+}\right]}{\partial x^2} + D_{Ca_y}\frac{\partial^2 \left[Ca^{2+}\right]}{\partial y^2} - u(t)\frac{\partial \left[Ca^{2+}\right]}{\partial x} - v(t)\frac{\partial \left[Ca^{2+}\right]}{\partial y} \tag{4}$$

For the sake of convenience we have taken $[Ca^{2+}] = $ C as calcium profile.

$$\frac{\partial C}{\partial t} = D_{Ca_x}\frac{\partial^2 C}{\partial x^2} + D_{Ca_y}\frac{\partial^2 C}{\partial y^2} - u(t)\frac{\partial C}{\partial x} - v(t)\frac{\partial C}{\partial y} \tag{5}$$

let

$$u = u_0 exp(-mt),\ v = v_0 exp(-mt) \tag{6}$$

Where u0 and v0 are initial velocity component along x and y axes respectively, C is the concentration at any time t in horizontal plane. A relation for steady and unsteady flow has been shown with exponentially or sinusoidal varying flow velocity as [12]

$$D_{Ca_x} = D_x = \alpha u \ and \ D_{Ca_y} = D_y = \alpha v \tag{7}$$

Where alpha is the coefficient having the dimension of length and

$$D_x = D_{x_0} exp(-mt),\ and \ D_y = D_{y_0} exp(-mt) \tag{8}$$

Where D_{x_0} and D_{y_0} are initial dispersion coefficient components along the two respective directions. The initial and boundary conditions are as given below

$$C = 0, t = 0, x \geq 0, y \geq 0 \tag{9}$$

$$C = C_0, t > 0, x = 0, y = 0 \tag{10}$$

The condition (9) is taken as the concentration is continuous across the inlet boundary and (10) indicate that there is no flux at the end of both boundaries. Also the change in calcium concentration is very negligible when x approaches to length l_1 and y approaches to length l_2 for $t > 0$, which we can write mathematically

$$\frac{\partial C}{\partial x} = 0, \frac{\partial C}{\partial y} = 0, t \geq 0, x \to l_1, y \to l_2 \tag{11}$$

Using (6) and (8) the differential equation (4) can be written as

$$\frac{1}{exp(-mt)} \frac{\partial C}{\partial t} = D_{x_0} \frac{\partial^2 C}{\partial x^2} + D_{y_0} \frac{\partial^2 C}{\partial y^2} - u_0 \frac{\partial C}{\partial x} - v_0 \frac{\partial C}{\partial y} \tag{12}$$

Now introducing the new time variable T by following transformation [6]

$$T = \int_0^t exp(-mt)dt = \frac{1}{m}[1 - exp(-mt)] \tag{13}$$

For an expression exp(-mt) which is taken such that exp(-mt)=1 for m=0 or t=0, the new time variable obtained from eq (13) satisfies the conditions T=0 for t=0 and T=infinity for m=0. Thus the nature of the initial condition does not change in the new time variable domain .Thus equation (12) can be written as

$$\frac{\partial C}{\partial T} = D_{x_0} \frac{\partial^2 C}{\partial x^2} + D_{y_0} \frac{\partial^2 C}{\partial y^2} - u_0 \frac{\partial C}{\partial x} - v_0 \frac{\partial C}{\partial y} \tag{14}$$

Let a new space variable is introduced as follows

$$X = x + y\sqrt{\frac{D_{y0}}{D_{x0}}} \tag{15}$$

Therefore differential equation (14) reduces into

$$\frac{\partial C}{\partial T} = D\frac{\partial^2 C}{\partial X^2} - U\frac{\partial C}{\partial X} \tag{16}$$

Where

$$D = D_{x_0}\left(1 + \frac{D_{y0}^2}{D_{x_0}^2}\right) : U = u_0 + v_0\sqrt{\frac{D_{y0}}{D_{x_0}}} \tag{17}$$

Using the transformations (13) and (15), the initial and boundary conditions (9)-(11) becomes as follows

$$C = 0, T = 0, X \geq 0 \tag{18}$$

$$C = C_0, T > 0, X = 0 \tag{19}$$

$$\frac{\partial C}{\partial X} = 0, T \geq 0, X \to \infty \tag{20}$$

Introducing a new dependent variable K(X,T) , by following transformation

$$C(X,T) = K(X,T)exp(\frac{U}{2D}X - \frac{U^2 T}{4D}) \tag{21}$$

Now applying Laplace Transform on Eqs. (16) and (18) - (20) and using initial condition (18), we get following ordinary boundary value problem.

$$\frac{d^2\overline{K}}{dX^2} = \frac{s}{D}\overline{K} \tag{22}$$

$$\overline{K} = \frac{C_0}{s - \frac{U^2}{4D}}, X = 0 \tag{23}$$

and

$$\frac{d\overline{K}}{dX} = 0, X \to \infty \tag{24}$$

Where $\overline{K}(X,s) = \int_0^\infty K(X,T)exp(-sT)dT$ and s is the Laplace parameter. Solution of (22) by using the conditions (22) and (24), becomes

$$\overline{K}(X,s) = \frac{C_0}{s - \frac{U^2}{4D}} exp\left(-X\sqrt{\frac{s}{D}}\right) \tag{25}$$

Applying inverse laplace transform on (25) and using (21) the solution is obtained for our problem

$$C(X,T) = \frac{1}{2}\left[erfC\left\{\frac{X-UT}{2\sqrt{DT}}\right\} + erfC\left\{\frac{X+UT}{2\sqrt{DT}}\right\}\right] \tag{26}$$

where

$$X = x + y\sqrt{\frac{D_{y_0}}{D_{x_0}}}, D = D_{x_0}\left(1+\frac{D_{y_0}^2}{D_{x_0}^2}\right) andU = u_0 + v_0\sqrt{\frac{D_{y_0}}{D_{x_0}}} \tag{27}$$

3 Results and Discussion

The solution (26) represents calcium concentration diffusion in cytosol, which represent the calcium concentration at distance x from mouth of the channel for any $t > 0$. The solution is in the form of complementary error function.

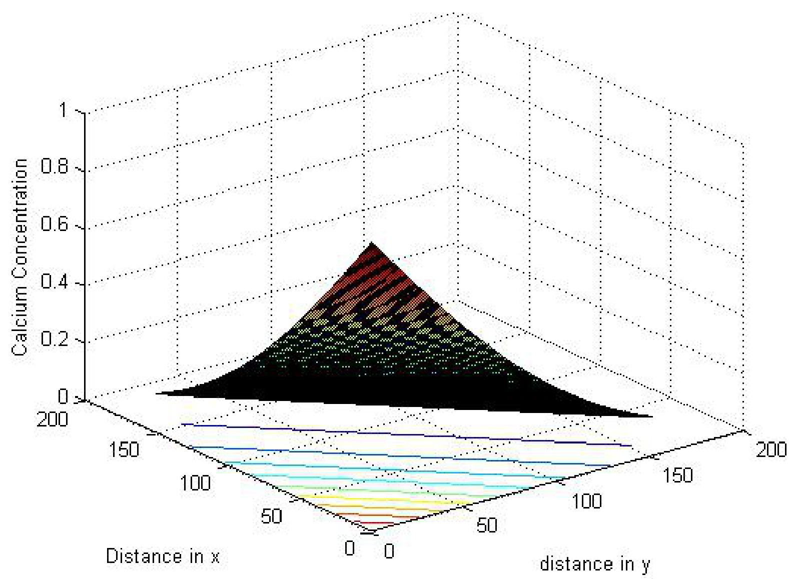

Figure 2: graph between calcium concentration and distance in x and y direction. $u = 10\mu m/s$, $v = 10\mu m/s$ Dx=250 and Dy-250 is taken

Figure 2 Shows the calcium concentration for any x and y for $t > 0$ which represent the calcium concentration decreasing uniformly as distance increasing. The calcium concentration is decreasing uniformly in both direction x and y form its initial value $.35\mu M$.

Figure 3 Shows calcium concentration is decreasing as y increasing when x is constant for any time $t > 0$ and it is observed that calcium concentration is linearly decreasing as y increasing for $t > 0$.

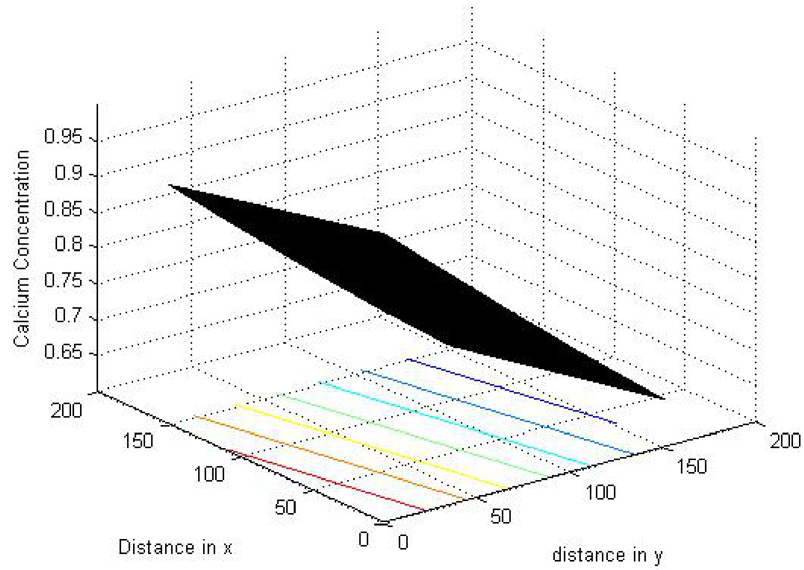

Figure 3: graph between calcium concentration and distance in x and y direction. $u = 0.1\mu m/s$, $v = 10\mu m/s$ Dx=20 and Dy-250 is taken

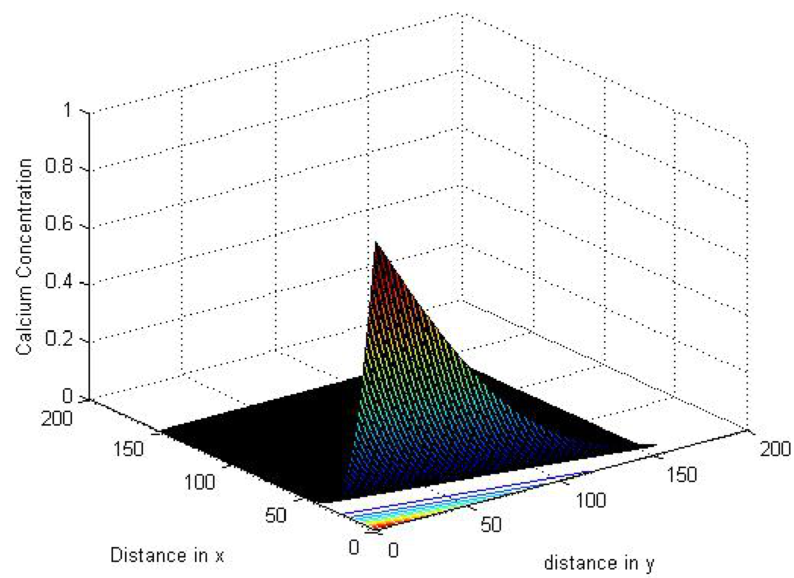

Figure 4: graph between calcium concentration and distance in x and y direction.$u = 10\mu m/s$, $v = 0.1\mu m/s$ Dx=250 and Dy-20 is taken

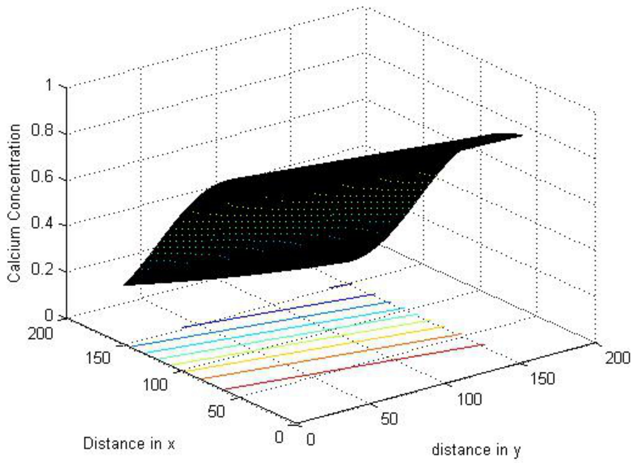

Figure 5: graph between calcium concentration and distance in x and y direction. $u = 10\mu m/s$, $v = 10\mu m/s$ Dx=250 and Dy=250 is taken for time t = 10s

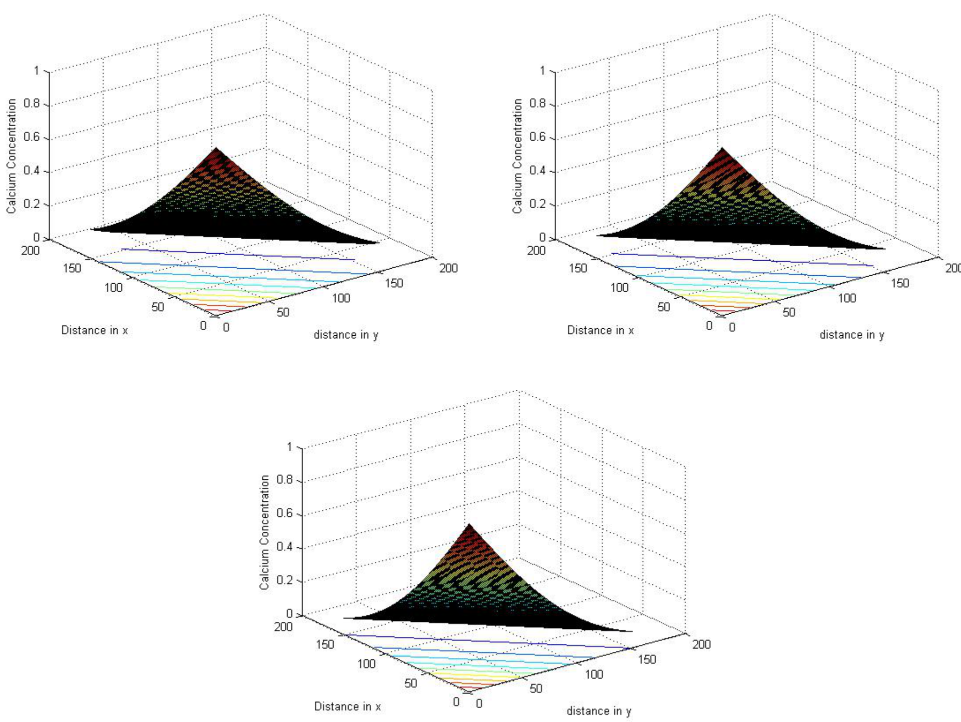

Figure 6: graph between calcium concentration and distance in x and y direction. with different values of diffusion coefficients

Figure 4 represents the calcium concentration is decreasing when x increasing for some fix value of y for $t > 0$. It is observed that when the value of x approaches to $.50\mu m$ the calcium concentration decreasing and then it becomes at steady state when $x > 50\mu m$ which is obvious in physical phenomena in calcium signalling in cytosol.

Figure 5 Shows the graph of calcium concentration for increasing value of time $t > 0$ for any fix value of x and y, which represent the effect of time on calcium concentration in cytosol and it is observed that calcium concentration is slowly decreasing after long time t for any fix x and y.

Figure 6 shows that the calcium concentration is decreasing for different value of diffusion coefficient for fix value of x, y and t. It is observed from the graph that when diffusion coefficient is increasing the calcium concentration is decreasing but when diffusion coefficient is very larger than the curve will depressed and calcium concentration is decreasing.

4 Conclusion

The mathematical modeling plays very important role for signal transduction in astrocytes. Advection diffusion is new invention in mathematical model for astrocytes cell. To incorporate more and more parameters in advection diffusion for future purpose of biophysical laboratory.

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