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Exact Solutions of the Klein-Gordon Equation by Modified Exp-Function Method[1]

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Abstract

In this paper, the modified exp-function method is used to seek generalized wave solutions of Klein-Gordon equation. As a result, some new types of exact traveling wave solutions are obtained which include kink wave solutions, periodic wave solution, and solitary wave solutions. Obtained results clearly indicate the reliability and efficiency of the proposed modified exp-function method.

Keywords: nonlinear wave equation, Klein-Gordon equation, exact wave solution,modified exp-function method

1 Introduction

The investigation of exact solutions of nonlinear wave equations plays an important role in the study of nonlinear physical phenomena. Recently, many effective methods for obtaining exact solutions of nonlinear wave equations have been proposed, such as bäcklund transformation method [1], homogeneous balance method [2,3], bifurcation method [4], Hirotas bilinear method [5], the hyperbolic tangent function expansion method [6,7], the Jacobi elliptic function expansion method [8,9], F-expansion method[10-12] and so on. He and Wu [13] developed the exp-function method to seek the solitary, periodic and compaction like solutions of nonlinear differential equations. It is an effective and simple method and is widely used. Based on this method, modified exp-function expansion method is proposed. The purpose of this paper is to find exact wave solutions of Klein-Gordon equation by the new method.

In this paper, we consider the Klein-Gordon equation

$$u_{tt} - u_{xx} + \alpha u + \beta u^3 = 0 \tag{1}$$

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where α, β are some nonzero parameters. The nonlinear Klein-Gordon equation appears in many types of nonlinearities. The Klein-Gordon equations play a significant role in many scientific applications such as solid state physics, nonlinear optics and quantum field theory, where $a \neq 0, b \neq 0$.

Many powerful methods, such as homotopy analysis method [18], the extend tanh method [19], the Exp-function method [20] were used to investigate these types of equations. The aim of this work is to further complement the studies on the the KleinCGordon equations.

2 Modified exp-function method

The exp-function method was first proposed by He and Wu to solve differential equations [13] and it was systematically studied in [14-17]. In this paper, we will introduce a modified exp-function method. The main procedures of this method are as follows.

We consider a general nonlinear PDE in the form

$$H(u, u_x, u_t, u_{xx}, u_{tt}, u_{tx}, \cdots) = 0 \tag{2}$$

Using a transformation

$$\xi = x - ct \tag{3}$$

where c are constants, we can rewrite Eq.(2) in the following nonlinear ODE:

$$H_1(u, u', u'', \cdots) = 0 \tag{4}$$

where the prime denotes the derivation with respect to ξ.
Let

$$u = v + s \tag{5}$$

where s are constants. Then Eq.(4) becomes

$$H_2(v, v', v'', \cdots) = 0 \tag{6}$$

Assume that the solution of Eq.(4) can be expressed in the following form

$$u(\xi) = \frac{\sum\limits_{i=-n}^{n} c_i g^i}{\sum\limits_{i=-n}^{n} d_i g^i} = \frac{\sum\limits_{i=0}^{2n} b_i g^i}{\sum\limits_{i=0}^{2n} a_i g^i} \tag{7}$$

where $g = e^{-k\xi}$ which is the solution of the homogeneous linear equation corresponding to equation (6), a_i, b_i are unknown to be further determined and n can be determined by homogeneous balance principle.

Substituting Eq.(7) into Eq.(4), we can get polynomial equation on g. Let the coefficient of g^i be zero, and solve the equation set, the a_i, b_i can be determined.

3 Solutions of Klein-Gordon equation

Using the transformation (3), equation (1) can be rewrite as

$$c^2u'' - u'' + \alpha u + \beta u^3 = 0 \tag{8}$$

Substituting Eq.(5) into Eq.(8), we have

$$(c^2 - 1)v'' + (v + 3\beta s^2)v + 3\beta s^2 v^2 + \beta v^3 + \beta s^3 + \alpha s = 0 \tag{9}$$

Let $s(\beta s^2 + \alpha) = 0$, then $s = 0$ or $s = \pm\sqrt{-\frac{\alpha}{\beta}}$.

According to homogeneous balance principle, we get $n = 1$.

case 1 $s = 0$.

The solution of the linear equation corresponding to equation (9) is

$$g = e^{-k\xi}, k = \sqrt{\frac{\alpha}{1 - c^2}} \tag{10}$$

So,we can assume

$$u(\xi) = \frac{\sum_{i=0}^{2n} b_i g^i}{\sum_{i=0}^{2n} a_i g^i} = \frac{b_0 + b_1 g + b_2 g^2}{a_0 + a_1 g + a_2 g^2} \tag{11}$$

Substituting Eq.(10) and Eq.(11) into Eq.(8) yields a set of algebraic equations for $g^i, i = 0, 1, \ldots, 6$. Letting the coefficients of these terms g^i to be zero yields a set of over-determined algebraic equations.

$$\begin{cases} \beta {b_0}^2 + \alpha b_0 {a_0}^2 = 0 \\ 3\alpha b_0 a_1 a_0 + 3\beta {b_0}^2 b_1 = 0 \\ 3\beta {b_0}^2 b_2 + 3\beta b_0 {b_1}^2 - 3\alpha b_2 {a_0}^2 + 3\alpha b_1 a_0 a_1 + 6\alpha b_0 a_2 a_0 = 0 \\ \alpha b_1 {a_1}^2 - \alpha b_0 a_1 a_2 + 6\beta b_0 b_1 b_2 + 8\alpha b_1 a_0 a_2 - \alpha b_2 a_0 a_1 + \beta {b_1}^3 = 0 \\ 3\alpha b_1 a_1 a_2 + 3\beta {b_1}^2 b_2 - 3\alpha b_0 {a_2}^2 + 6\alpha b_2 a_0 a_2 + 3\beta b_0 {b_2}^2 = 0 \\ 3\beta b_1 {b_2}^2 + 3\alpha b_2 a_1 a_2 = 0 \\ \alpha b_2 {a_2}^2 + \beta {b_2}^3 = 0 \end{cases}$$

Solving the system of algebraic equations by use of Maple, we obtain

$$a_0 = -\frac{\beta {b_1}^2}{8\alpha a_2}, a_1 = 0, a_2 = a_2, b_0 = 0, b_1 = b_1, b_2 = 0 \tag{12}$$

where $a_2 \neq 0, b_1$ is an arbitrary constant.

Substituting (12) into (11), we obtain following solutions of the Klein-Gordon equation

$$u(x,t) = \mathrm{u}(\xi) = \frac{b_1}{-\frac{\beta {b_1}^2}{8\alpha a_2} g^{-1} + a_2 g} \tag{13}$$

where $g = e^{-\sqrt{\frac{\alpha}{1-c^2}}\xi}$, $\xi = x - ct$.

case 1.1 Let $b_1 = 2, a_2 = \pm\sqrt{\frac{-\beta}{2\alpha}}$, $\alpha \cdot \beta < 0$.

when $\frac{\alpha}{1-c^2} > 0$, we have

$$u_{1,2}(x,t) = \pm\sqrt{\frac{-2\alpha}{\beta}} \sec h(\sqrt{\frac{\alpha}{1-c^2}}(x-ct)) \tag{14}$$

when $\frac{\alpha}{1-c^2} < 0$, we have

$$u_{3,4}(x,t) = \pm\sqrt{\frac{-2\alpha}{\beta}} \sec(\sqrt{\frac{\alpha}{c^2-1}}(x-ct)) \tag{15}$$

case 1.2 Let $b_1 = 2, a_2 = \pm\sqrt{\frac{\beta}{2\alpha}}$, $\alpha \cdot \beta > 0$.

when $\frac{\alpha}{1-c^2} > 0$, we have

$$u_{5,6}(x,t) = \pm\sqrt{\frac{2\alpha}{\beta}} \csc h(\sqrt{\frac{\alpha}{1-c^2}}(x-ct)) \tag{16}$$

when $\frac{\alpha}{1-c^2} < 0$, we have

$$u_{7,8}(x,t) = \pm I\sqrt{\frac{2\alpha}{\beta}} \csc(\sqrt{\frac{\alpha}{c^2-1}}(x-ct)) \tag{17}$$

where $I^2 = -1$.

case 2 $s = \pm\sqrt{-\frac{\alpha}{\beta}}$.

The solution of the linear equation corresponding to equation (9) is

$$g = e^{-k\xi}, k = \sqrt{\frac{2\alpha}{c^2-1}}. \tag{18}$$

So,we can assume

$$\phi(\xi) = \frac{\sum_{i=0}^{2n} b_i g^i}{\sum_{i=0}^{2n} a_i g^i} = \frac{b_0 + b_1 g + b_2 g^2}{a_0 + a_1 g + a_2 g^2}. \tag{19}$$

Substituting Eq.(18) and Eq.(19) into Eq.(8) yields a set of algebraic equations for $g^i, i = 0, 1, \ldots, 6$. Letting the coefficients of these terms g^i to be zero yields a set of over-determined algebraic equations.

$$\begin{cases} \beta {b_0}^3 + \alpha b_0 {a_0}^2 = 0 \\ 3\beta {b_0}^2 b_1 + 3\alpha b_1 {a_0}^2 = 0 \\ 9\alpha b_2 {a_0}^2 + 3\beta {b_0}^2 b_2 + 3\beta b_0 {b_1}^2 + 3\alpha b_0 {a_1}^2 - 6\alpha b_0 a_2 a_0 = 0 \\ 8\alpha b_0 a_1 a_2 + \alpha b_1 {a_1}^2 + \beta {b_1}^3 + 8\alpha b_2 a_0 a_1 + 6\beta b_0 b_1 b_2 - 10\alpha b_1 a_0 a_2 = 0 \\ 3\alpha b_2 {a_1}^2 + 3\beta {b_1}^2 b_2 - 6\alpha b_2 a_0 a_2 + 3\beta b_0 {b_2}^2 + 9\alpha b_0 {a_2}^2 = 0 \\ 3\beta b_1 {b_2}^2 + 3\alpha b_1 {a_2}^2 = 0 \\ \beta {b_2}^3 + \alpha b_2 {a_2}^2 = 0 \end{cases}$$

Solving the system of algebraic equations by use of Maple, we obtain

$$a_0 = \pm \frac{\alpha {a_1}^2 + {b_1}^2 \beta}{4\alpha \sqrt{-\frac{\beta}{\alpha}} b_2}, a_2 = \pm \sqrt{-\frac{\beta}{\alpha}} b_2, a_1 = a_1, b_0 = \frac{\alpha {a_1}^2 + {b_1}^2 \beta}{4 b_2 \beta}, b_1 = b_1, b_2 = b_2. \tag{20}$$

Where $b_2 \neq 0$, a_1, b_1 are arbitrary constants.

Substituting (20) into (19), we obtain following solutions of the Klein-Gordon equation

$$u(x,t) = \frac{\frac{\alpha {a_1}^2 + {b_1}^2 \beta}{4 b_2 \beta} + b_1 g + b_2 g^2}{\pm \frac{\alpha {a_1}^2 + {b_1}^2 \beta}{\sqrt{-\frac{\beta}{\alpha}} 4\alpha b_2} + a_1 g \pm \sqrt{-\frac{\beta}{\alpha}} b_2 g^2}. \tag{21}$$

where $g = e^{-\sqrt{\frac{2\alpha}{c^2-1}}\xi}, \xi = x - ct$.

case 2.1 Let $b_1 = b_2 = 1, a_1 = \pm\sqrt{\frac{-\beta}{\alpha}}$, $\alpha \cdot \beta < 0$.

When $\frac{\alpha}{c^2-1} > 0$, we have

$$u_{9,10}(x,t) = \pm\sqrt{\frac{-\alpha}{\beta}} \coth(\sqrt{\frac{\alpha}{2(c^2-1)}}(x - ct)). \tag{22}$$

When $\frac{\alpha}{c^2-1} < 0$, we have

$$u_{11,12}(x,t) = \pm I \sqrt{\frac{-\alpha}{\beta}} \cot(\sqrt{\frac{\alpha}{2(1-c^2)}}(x - ct)), \tag{23}$$

where $I^2 = -1$.

case 2.2 Let $b_1 = 1, b_2 = -1, a_1 = \pm\sqrt{\frac{-\beta}{\alpha}}$, $\alpha \cdot \beta < 0$.

When $\frac{\alpha}{c^2-1} > 0$, we have

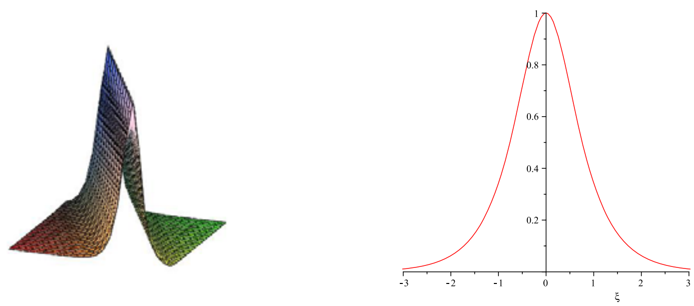
$$u_{13,14}(x,t) = \pm\sqrt{\frac{-\alpha}{\beta}} \tanh(\sqrt{\frac{\alpha}{2(c^2-1)}}(x - ct)). \tag{24}$$

When $\frac{\alpha}{c^2-1} < 0$, we have

$$u_{15,16}(x,t) = \sqrt{\frac{-\alpha}{\beta}}\tan\left(\sqrt{\frac{\alpha}{2(1-c^2)}}(x-ct)\right). \tag{25}$$

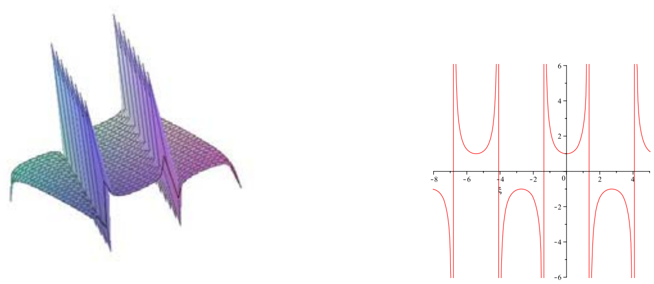
In order to well understand the solutions that we have got, some typical figures are given as follows.

1. $u_{1,2}$ are solitary wave solutions.



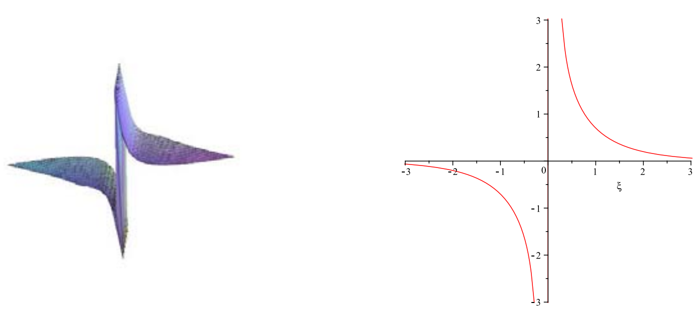
(1-1) two-dimensional figure (1-2) three-dimensional figure
Fig.1 $u_{1,2}$ for $\alpha = -1, \beta = 2, c = 2$.

2. $u_{3,4}$ are periodic solitary wave solutions.



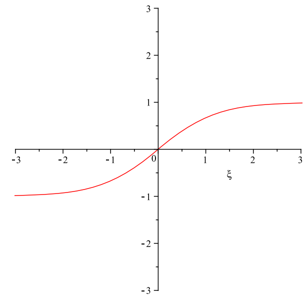
(2-1) two-dimensional figure (2-2) three-dimensional figure
Fig.2 $u_{3,4}$ for $\alpha = -1, \beta = 2, c = 1/2$.

3. $u_{5,6}$ are kink wave solutions.



(3-1) two-dimensional figure (3-2) three-dimensional figure
Fig.3 $u_{5,6}$ for $\alpha = 1, \beta = 2, c = 1/2$.

4. $u_{13,14}$ are kink wave solutions.



(4-1) two-dimensional figure (4-2) three-dimensional figure
Fig.4 $u_{13,14}$ for $\alpha = -1, \beta = 2, c = 1/2$.

4 Conclusions

In this paper, we have obtained some new solitary solutions of the Klein-Gordon equation (1) by using the modified exp-function method. It shows that the new method is powerful and straightforward for nonlinear differential equations. It is said that this method can be applied to other kinds of nonlinear problems.

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