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A New Approach to Improved Multiquadric Quasi-Interpolation by Using General Hermite Interpolation

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Abstract

In this paper, a new approach to improve univariate multiquadric operators is surveyed. The presented scheme is obtained by using Hermite interpolating polynomials where the function is approximated by generalized L_B quasi-interpolation operator. Error analysis shows that the convergence rate depends on the shape parameter c. Thus, our operators could provide the desired smoothness and precision by choosing the suitable value of c. The advantage of the resulting scheme is that the algorithm is simple and provides a high degree of accuracy.

Keywords: Hermite interpolating polynomials, Quasi-interpolation, Convergence rate, Multiquadrics

1 Introduction

Hrady [8] proposed multiquadric (MQ) in 1968 as a kind of radial basis function (RBF). In 1992, Beatson and Powel [1] proposed three univariate multiquadric quasi-interpolation. They named them L_A, L_B, L_C to approximate a function $f : [a, b] \to \mathbb{R}$ on the scattered points $a = x_0 < x_1 < \cdots < x_N = b$. Afterwards, Wu and Schaback [12] proposed a multiquadric quasi interpolation L_D, which possesses shape preserving and linear reproducing on $[x_0, x_N]$. They proved that when the shape parameter $c = O(h)$, where h is the maximum distance between adjacent centers, the error of the operator L_D is $O(h^2|\ln h|)$.

Recently many works have been done on this subject. Ling [10] proposed a multilevel MQ operator using the operator L_D, and proved that it converges with a rate of $O(h^{2.5}|\ln h|)$ as $c = O(h)$. Feng & Li [7] constructed a

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shape-preserving quasi-interpolation operator by shifts of qubic multiquadrics. They showed that the operator satisfies the quadric polynomial reproduction property and produces an error of $O(h^2)$ as $c = O(h)$. Furthermore, many researchers provided some examples using multiquadric quasi-interpolation to solve differential equations [3, 4, 5, 6, 9].

The aim of our paper is to present multiquadric quasi-interpolation operators with higher accuracy. Based on [11], which the authors proposed quasi-interpolation operators $\mathrm{L}_{H_{2m-1}}$, we propose a kind of improved quasi-interpolation operators $\mathrm{L}_{H_{3m-1}}$, by combining the operator L_B with Hermite interpolating polynomials. We show that the new operators could reproduce polynomials of higher degree. Our analysis indicates that the convergence rate depends heavily on c. Thus, our operators could provide the desired smoothness and precision by choosing the suitable value of c.

The rest of the paper is organized as follows: In Section 2, we define the improved multiquadric quasi-interpolation operators $\mathrm{L}_{H_{3m-1}}$. Afterwards, we obtain error analysis. In Section 3, two examples for testing our method is showed and in the last section the conclusion is derived.

2 The improved quasi-interpolation operators by using Hermite interpolating polynomials

In this section, we first define the improved quasi-interpolation operators $\mathrm{L}_{H_{3m-1}}$, then give our main results including the polynomial reproduction property and convergence rate.

The quasi-interpolation operator L_B is defined as follows

$$(\mathrm{L}_B f)(x) = f(x_0)\,\psi_0(x) \,+\, \sum_{i=1}^{N-1} f(x_i)\,\psi_i(x) \,+\, f(x_N)\,\psi_N(x)\,, \quad x \in [a,\, b], \quad (1)$$

where

$$\begin{aligned}\psi_0(x) &= \frac{1}{2}\,c^2 \int_{-\infty}^{x_0} \frac{1}{[\,(x-\theta)^2 \,+\, c^2\,]^{3/2}}\,\mathrm{d}\theta \,+\, \frac{1}{2}\,c^2 \int_{x_0}^{x_1} \frac{(x_1-\theta)/\,(x_1-x_0)}{[\,(x-\theta)^2 \,+\, c^2\,]^{3/2}}\,\mathrm{d}\theta \\ &= \frac{1}{2} \,+\, \frac{\varphi_1(x) \,-\, \varphi_0(x)}{2\,(x_1 \,-\, x_0)}, \qquad (2)\end{aligned}$$

$$\begin{aligned}\psi_N(x) &= \frac{c^2}{2}\Big(\int_{x_N}^{\infty} \frac{1}{[\,(x-\theta)^2 \,+\, c^2\,]^{3/2}}\,\mathrm{d}\theta + \int_{x_{N-1}}^{x_N} \frac{(\theta-x_{N-1})/\,(x_N-x_{N-1})}{[\,(x-\theta)^2 \,+\, c^2\,]^{3/2}}\,\mathrm{d}\theta\Big) \\ &= \frac{1}{2} \,-\, \frac{\varphi_N(x) \,-\, \varphi_{N-1}(x)}{2\,(x_N \,-\, x_{N-1})}, \qquad (3)\end{aligned}$$

$$\psi_i(x) = \frac{1}{2}\, c^2 \int_{x_{i-1}}^{x_{i+1}} \frac{B_i(\theta)}{[\,(x-\theta)^2 \,+\, c^2\,]^{3/2}}\, \mathrm{d}\theta$$
$$= \frac{\varphi_{i+1}(x) \,-\, \varphi_i(x)}{2\,(x_{i+1} \,-\, x_i)} - \frac{\varphi_i(x) \,-\, \varphi_{i-1}(x)}{2\,(x_i \,-\, x_{i-1})}, \tag{4}$$

and

$$\varphi_i(x) = \sqrt{(x - x_i)^2 \,+\, c^2}, \qquad c > 0. \tag{5}$$

for $i = 1, 2, \ldots, N-1$, where $B_i(\theta)$ is the piecewise linear hat function having the knots $\{x_{i-1}, x_i, x_{i+1}\}$, and satisfying $B_i(x_i) = 1$.

Here, we extend the proposed method in [11], and define the improved quasi-interpolation operators $\mathrm{L}_{H_{3m-1}}$ as follows

$$(\mathrm{L}_{H_{3m-1}} f)(x) = \sum_{i=0}^{N} \psi_i(x)\, H_{3m-1}[f; x_{i-1}, x_i, x_{i+1}](x), \tag{6}$$

where $H_{3m-1}[f; x_{i-1}, x_i, x_{i+1}](x)$ are general Hermite interpolating polynomials of degree $3m-1$ which agree with the function f at the points

$$\underbrace{x_{i-1}, x_{i-1}, \ldots, x_{i-1}}_{m}\, \underbrace{x_i, x_i, \ldots, x_i}_{m},\, \underbrace{x_{i+1}, x_{i+1}, \ldots, x_{i+1}}_{m}.$$

The quasi-interpolation $(\mathrm{L}_{H_{3m-1}} f)(x)$ are C^∞ function on $[a,\, b]$, and the operators $\mathrm{L}_{H_{3m-1}}$ have the following polynomial reproduction property.

Theorem 1. *The operators* $\mathrm{L}_{H_{3m-1}}$ *reproduce all polynomials of degree* $\leqslant 3m-1$.

Proof. It is well known that

$$H_{3m-1}[f; x_{i-1}, x_i, x_{i+1}](x) = f(x), \qquad \forall f \in \mathbb{P}_{3m-1}, \tag{7}$$

where $\mathbb{P}_{3m-1}$ denotes the set of all polynomials of degree $\leqslant 3m-1$. Thus, we have

$$\mathrm{L}_{H_{3m-1}} f \equiv f, \qquad \forall f \in \mathbb{P}_{3m-1}, \tag{8}$$

i.e., $\mathrm{L}_{H_{3m-1}}$ reproduce all polynomials of degree $\leqslant 3m-1$. $\square$

According to [11], we define these notations for considering convergence rate of the operators,

$$I_\rho(x) = [x-\rho,\, x+\rho], \quad \rho > 0,$$

$$h = \inf\{\rho > 0 : \ \forall x \in [a,\, b],\ I_\rho(x) \cap X \neq \emptyset\},$$

$$M = \max_{x \in [a,\, b]} \#\,(I_h(x) \cap X),$$

where $X = \{x_0, x_1, \ldots, x_N\}$ and $\#(\cdot)$ denotes the cardinality function. It is easy to check that $2h = \max\{\mid x_1 - x_0 \mid, \mid x_2 - x_1 \mid, \ldots, \mid x_N - x_{N-1} \mid\}$, and M is the maximum number of points of X contained in an interval $I_h(x)$.

Theorem 2. *Assume that the shape parameter c satisfies*

$$c \leqslant Dh^l,$$

where D is a positive constant, and l is a positive integer. If $f \in C^{3m}([a,\, b])$, then

$$\|\mathrm{L}_{H_{3m-1}} f - f\|_\infty \leqslant KM \|f^{(3m)}\|_\infty \varepsilon_{l,m}(h), \tag{9}$$

where $\|\cdot\|_\infty$ denotes the sup-norm on $[a,\, b]$,

$$\varepsilon_{l,m}(h) = \begin{cases} h^{3m}, & \text{if } 3m < 2l-1, \\ h^{2l-1}, & \text{if } 3m \geqslant 2l-1, \end{cases} \tag{10}$$

and K is a positive constant independent of x and X.

Proof. The proof is similar to theorem 2 in [11]. $\square$

3 Numerical examples

In this section, we consider the following functions on [0, 1], which these functions are given in [12]

$$\text{Saddle} \qquad f_1 = \frac{1.25}{6 + 6(3x-1)^2}, \tag{11}$$

$$\text{Sphere} \qquad f_2 = \frac{\sqrt{64 - 81(x-0.5)^2}}{9} - 0.5. \tag{12}$$

We apply the operators $\mathrm{L}_{H_{3m-1}}$ and $\mathrm{L}_{H_{2m-1}}$ with $c = (2h)^l$ on both of these functions.

The numerical results using uniform grids of 21 points for the operators $\mathrm{L}_{H_{3m-1}}$ and $\mathrm{L}_{H_{2m-1}}$ are given in Tables 1 and 2. In order to compare these methods, we calculate the approximating functions at the points $\frac{i}{101}$, $i = 1, \ldots, 100$. Tables 1 and 2 show the mean and maximum errors which are calculated for different values of the parameters l and m. The numerical results show that the improved quasi-interpolation operators $\mathrm{L}_{H_{3m-1}}$ have good approximating behavior.

	$L_{H_{3m-1}}f_1$		$L_{H_{2m-1}}f_1$	
(l, m)	ε_{mean}	ε_{max}	ε_{mean}	ε_{max}
(2, 1)	0.1640×10^{-4}	0.5783×10^{-4}	0.2654×10^{-3}	0.1200×10^{-2}
(2, 2)	0.3087×10^{-5}	0.1149×10^{-4}	0.6678×10^{-5}	0.4005×10^{-4}
(3, 1)	0.1708×10^{-4}	0.6072×10^{-4}	0.2541×10^{-3}	0.1180×10^{-2}
(3, 2)	0.3755×10^{-7}	0.2777×10^{-6}	0.6435×10^{-5}	0.4160×10^{-4}
(4, 1)	0.1708×10^{-4}	0.6073×10^{-4}	0.2540×10^{-3}	0.1180×10^{-2}
(4, 2)	0.3301×10^{-7}	0.2768×10^{-6}	0.6435×10^{-5}	0.4160×10^{-4}

Table 1. Numerical results for the saddle function.

	$L_{H_{3m-1}}f_2$		$L_{H_{2m-1}}f_2$	
(l, m)	ε_{mean}	ε_{max}	ε_{mean}	ε_{max}
(2, 1)	0.2422×10^{-5}	0.1202×10^{-4}	0.3037×10^{-3}	0.5727×10^{-3}
(2, 2)	0.1474×10^{-5}	0.3126×10^{-5}	0.1590×10^{-5}	0.4065×10^{-5}
(3, 1)	0.2307×10^{-5}	0.1232×10^{-4}	0.2848×10^{-3}	0.5653×10^{-3}
(3, 2)	0.4669×10^{-8}	0.1544×10^{-7}	0.5714×10^{-6}	0.3077×10^{-5}
(4, 1)	0.2307×10^{-5}	0.1232×10^{-4}	0.2848×10^{-3}	0.5653×10^{-3}
(4, 2)	0.1024×10^{-8}	0.7906×10^{-8}	0.6017×10^{-6}	0.3107×10^{-5}

Table 2. Numerical results for the sphere function.

4 Conclusions

In this paper, a kind of improved multiquadric quasi-interpolation operators is proposed. The operators reproduce polynomials of higher degree. Under a certain assumption, a result on the convergence rate of the operators is given. The numerical examples show that proposed method provides a high degree of accuracy.

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