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Nonlinear Piecewise Defined Difference Equations

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Abstract

In this paper we determine periodicity of orbits of a piecewise defined difference equation, which have a quadratic term.

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1 Introduction

We define here a new kind of sequences, which represents a nonlinear difference equation (recurrence relations). There are two rules for the relation between the successive terms. One of the rules contain a nonlinear term. Precisely, we start with

Definition Let h, j, v and Q_0 be positive integers. We compute the sequence Q_k as follows

$Q_k = hQ_{k-1}^2$, if $Q_{k-1} \leq v$,
$Q_k = hQ_{k-1} - j$, if $Q_{k-1} > v$, for $k = 1, 2, ...$

For example, the following setting $h = 2, j = 2, v = 1$, $Q_0 = 1$ yields the sequence

$$1,\ 2\ ,\ 2,\ ...$$

In general for any Q_0 and h the choice

$$j = Q_0^2 h^2 - Q_0^2 h, Q_0 \leq v < Q_0^2 h$$

leads us to the sequence

$$Q_0, Q_0^2 h, Q_0^2 h, Q_0^2 h, ...$$

Also, it is easy to see that for any positive integer i the choice

$$Q_0 = 1,\ h > 1,\ j = h^l \text{ and } h^{k-1} \leq v < h^{l-1} - 1 \text{ for } k = 2^{i-1}\ ,\ l = 2^i$$

then the sequence takes the form

$$1, h, h^3,\ ...,\ h^{l-1}, 0, 0,\ ...$$

We see from this example that we can get some fixed or in general periodic behavior of the sequence after a certain term. We are interested here in determining conditions, which guarantees this behavior.

2 Periodic sequences

In this section we consider sequences, which has periodic behavior after certain terms. We assume that $Q_0 = 1$ in the further considerations. The simplest case is that we have a constant sequence after a certain term. When we require that

$$j = h^l - h^{l-1}, h^k \leq v < h^{l-1} \text{ for a positive integer } l$$

then we obtain the sequence

$$1, h, h^3, \ldots, h^{l-1}, h^{l-1}, \ldots$$

Now, we look for more complicated behavior. There are cases, in which the periodic subsequences decreases monotonically from the first term to the last term. The following proposition illustrates this case

Proposition 1: Let h, i, n and t be integers such that $h \geq 2$, $t \geq 1$ and $0 \leq n \leq i$. We define positive integers

$$m = 2^n - 1, k = 2^{i-1} - 1, l = 2^i, j = \frac{(h^{l+t} \pm h^m)(h-1)}{h^{t+1} - 1}$$

If we choose any integer v such that

$$h^k \leq v \leq h^{l+t-1} - j\frac{h^t - 1}{h-1} - 1$$

then $Q_{i+t+1} = \mp Q_n$.

Example: We set $i = 3$, $n = 0$, $t = 6$ and choose the minus sign. The triplet $h = 2, j = 129, v \in [8, 64]$ yields

$$1, 2, 8, 128, 127, 125, 121, 113, 97, 65, 1, 2, \ldots$$

Example: We set $i = 3$, $n = 1$, $t = 5$ and choose the minus sign. The triplet $h = 2, j = 130, v \in [8, 65]$ yields

$$1, 2, 8, 128, 126, 122, 114, 98, 66, 2, \ldots$$

Example: We set $i = 1, n = 0, t = 1$ and choose the plus sign. The triplet $h = 3, v = 1, j = \frac{h^3+1}{h+1} = \frac{(h^3+1)(h-1)}{h^2-1} = 7$ yields

$1, 3, 2, -1, 3, \ldots$

Proof. We note first that

$Q_0 = 1, Q_1 = h, Q_2 = h^3, \ldots, Q_{i-1} = h^k \leq v, Q_i = h^{l-1}$,

Now, according to our assumptions

$$\mp h^m \leq h^{l-1} \ldots\ldots.(1)$$

On the other hand, $h^l - j = h^l - \frac{(h^{l+t} \pm h^m)(h-1)}{h^{t+1}-1} = \frac{1}{h^{t+1}-1}[h^{l+t} - h^l \mp h^m(h-1)]$

Now,

$h^l - j \leq h^{l-1}$ iff $-h^l \mp h^m(h-1) \leq -h^{l-1}$ iff $\mp h^m(h-1) \leq h^{l-1}(h-1)$

Due to (1) it is true. Hence, we have

$$h^{l-1} \geq h^l - j.......(2)$$

Upon applying (2) twice we obtain

$Q_i = h^{l-1} \geq h^l - j = hh^{l-1} - j \geq h(h^l - j) - j$

Again the application of (2) several times yields

$Q_i \geq h(h(...(h(h^l - j) - j)...) - j) - j$

In fact the application of (2) t times yields

$Q_i \geq h^{l-1+t} - j\frac{h^t-1}{h-1}$

Hence, $Q_i > v$. Moreover, $Q_{i+1} = hQ_i - j = hh^{l-1} - j$.

We apply (2) once again $t-1$ times and obtain

$Q_{i+1} \geq h(h^l - j) - j \geq ... \geq h(h(...(h(h^l - j) - j)...) - j) - j = h^{l-1+t} - j\frac{h^t-1}{h-1}$

Hence, $Q_{i+1} > v$. In this manner we can prove that

$Q_{i+g} > v \quad \forall g \in \{1, 2, ..., t\}$

This means that

$Q_{i+g} = hQ_{i+g-1} - j \quad \forall g \in \{1, 2, ..., t+1\}$

We are interested in Q_{i+t+1}, which is

$h(h(...(h(Q_i - j) - j)...) - j) - j = h(h(...(h(h^{l-1} - j) - j)...) - j) - j$

$= h^{l-1+t+1} - j\frac{h^{t+1}-1}{h-1} = \mp h^m$ ■

Using the computer we find that all the possible integer values for $j < 300$, which leads to periodic sequences, are

$$3, 5, 9, 10, 11, 14, 15, 17, 18, 128, 129, 130, 136, 146, 168, 170, 171, 248, 254, 255, 257, 258, 264, 266$$

The assignment $j = \frac{(h^{l+t}+h^m)(h-1)}{h^{t+1}-1}$ leads to

n	p	t	j	n	p	t	j	n	p	t	j	n	p	t	j
0	2	0	15	0	3	3	$\frac{2047}{15}$	1	3	0	254	2	3	0	264
0	3	0	255	1	3	3	$\frac{682}{5}$	1	2	2	$\frac{62}{7}$	1	1	1	2
0	2	2	9	1	3	1	170	1	2	1	10	1	0	0	0
0	2	1	$\frac{31}{3}$	1	3	2	146	1	2	0	14	1	0	1	$\frac{2}{3}$

and the assignment $j = \frac{(h^{l+t}-h^m)(h-1)}{h^{t+1}-1}$ leads to

n	p	t	j	n	p	t	j	n	p	t	j
0	1	0	5	1	3	0	258	0	3	6	129
1	1	0	6	0	3	0	257	1	3	5	130
0	2	2	$\frac{65}{7}$	0	3	1	171	2	3	3	136
0	2	1	11	1	2	0	18	2	3	1	168

Hence, we explained by the previous results all values except $j = 266$. The setting $h = 2, j = 266, v \in [8, 127]$ yields

$$1, 2, 8, 128, -10, 200, 134, 2, 8, ...$$

We notice that this sequence includes periodic subsequences starting from an element and going back (not in a monotone manner) to an existing element.

We will use the following notation
$G(q) = 1+2h+...+(q-2)h^{q-3}+(q-3)h^{q-2}+...+h^{2q-6} = \sum_{w=0}^{q-3}(w+1)h^w + \sum_{w=1}^{q-3} wh^{2q-w-5}$,
$F(p, q) = h^{2q-2p-4}G(p + 3), K(p, q) = h^q + h^{q+1} + ... + h^{q+p}$.

We note that

$$K(q-2, 0)^2 - 2h^{q-2}K(q-2, 0) = G(q) - h^{2q-4} \text{ ... (1)}$$
$$K(q-2, 0)^2 = G(q + 1) \text{ ... (2)}$$

Lemma 1: $F(p, q) = F(p-1, q) + 2K(p, 2q-2p-4) - h^{2q-2p-4}$ for every natural numbers $p < q$.

Proof. According to definition
$G(p + 2) = \sum_{w=0}^{p-1}(w + 1)h^w + \sum_{w=1}^{p-1} wh^{2p-w-1}$
$F(p-1, q) + 2K(p, 2q-2p-4) - h^{2q-2p-4} =$
$h^{2q-2p-2}G(p + 2) + 2K(p, 2q-2p-4) - h^{2q-2p-4} =$
$h^{2q-2p-2}(\sum_{w=0}^{p-1}(w + 1)h^w + \sum_{w=1}^{p-1} wh^{2p-w-1}) + 2K(p, 2q-2p-4) - h^{2q-2p-4} =$

$\sum_{w=0}^{p-1}(w + 1)h^{2q-2p-2+w} + \sum_{w=1}^{p-1} wh^{2q-w-3} + 2\sum_{w=2q-2p-4}^{2q+p-4} h^w - h^{2q-2p-4} =$

$\sum_{w=0}^{p-1}(w + 1)h^{2q-2p-2+w} + \sum_{w=1}^{p-1} wh^{2q-w-3} + 2\sum_{w=2q-2p-3}^{2q+p-4} h^w + h^{2q-2p-4} =$

$\sum_{w=0}^{p-1}(w + 1)h^{2q-2p-2+w} + \sum_{w=1}^{p-1} wh^{2q-w-3} + 2\sum_{w=0}^{p-1} h^{2q-2p-3+w} + h^{2q-2p-4} =$

$\sum_{w=1}^{p} wh^{2q-2p-3+w} + \sum_{w=1}^{p-1} wh^{2q-w-3} + 2\sum_{w=0}^{p-1} h^{2q-2p-3+w} + h^{2q-2p-4} =$

$\sum_{w=1}^{p-1}(w+2)h^{2q-2p-3+w}+ph^{2q-p-3}+\sum_{w=1}^{p-1}wh^{2q-w-3}+2h^{2q-2p-3}+h^{2q-2p-4}=$

$\sum_{w=0}^{p}(w+1)h^{2q-2p-4+w}+ph^{2q-p-3}+\sum_{w=1}^{p-1}wh^{2q-w-3}=$

$\sum_{w=0}^{p}(w+1)h^{2q-2p-4+w}+\sum_{w=1}^{p}wh^{2q-w-3}=$

$=h^{2q-2p-4}(\sum_{w=0}^{p}(w+1)h^{w}+\sum_{w=1}^{p}wh^{2p-w+1})=h^{2q-2p-4}G(p+3)=F(p,q)$ ■

Lemma 2: $G(q+1)>F(y,q)$ for $0<y<q-2$.

Proof. According to lemma 1 we know that is an increasing function in the first argument. Hence we have only to check the inequality for $y=q-3$. Now,

$F(q-3,q)=h^2G(q)=h^2(1+2h+...+(q-2)h^{q-3}+(q-3)h^{q-2}+...+h^{2q-6})=$
$h^2+2h^3+...+(q-2)h^{q-1}+(q-3)h^{q}+...+h^{2q-4})<1+2h+3h^2+...+(q-1)h^{q-2}+$
$(q-2)h^{q-1}+...+h^{2q-4}$ ■

Lemma 3: $G(q-u+3)-2K(q-2,q-u)-F(u-3,q)=G(q-u+2)-F(u-2,q)$ for $u\geq 3$.

Proof. According to definition we obtain

$G(q)-G(q-1)=1+2h+...+(q-2)h^{q-3}+(q-3)h^{q-2}+...+h^{2q-6}-$
$(1+2h+...+(q-3)h^{q-4}+(q-4)h^{q-3}+...+h^{2q-8})=$
$2h^{q-3}+2h^{q-4}+...+2h^{2q-8}+2h^{2q-7}+h^{2q-6}=2K(q-4,q-3)+h^{2q-6}$

Thus,

$G(q-u+3)-G(q-u+2)=2K(q-u-1,q-u)+h^{2q-2u}$

From this we deduce that

$G(q-u+3)-2K(q-2,q-u)-F(u-3,q)=$
$G(q-u+2)+2K(q-u-1,q-u)+h^{2q-2u}-2K(q-2,q-u)-F(u-3,q)=$
$G(q-u+2)-2K(u-2,2q-2u)+h^{2q-2u}-F(u-3,q)$

According to lemma 1 by setting $p=u-2$ we obtain

$F(u-2,q)=F(u-3,q)+2K(u-2,2q-2u)-h^{2q-2u}$ ■

We give now the result, which generalizes the previous example

Proposition 2: Let $h,i,m>1$, $q>2$. If we choose $k=2^{i-1}-1, l=2^{i}$,

$j=\frac{1}{2h^q}\left(K(q-2,0)+2h^{q+l}\pm\sqrt{4h^qh^m+G(q+1)+4K(q-2,q+l)}\right),$

then for every v such that $h^k \leq v < h^{l-1}$ the sequence takes the form
$1, h, h^3, ..., h^{l-1}, Q_{i+1} < v, Q_{i+2} > v, ..., Q_{q+i} > v, Q_{q+i+1} = h^m, ...$
Proof: It is trivial to prove that $Q_i = h^{l-1}$. Let $R = \sqrt{4h^q h^m + G(q+1) + 4K(q-2, q+l)}$
Now, $j = \frac{1}{2h^q}\left(K(q-2,0) + 2h^{q+l} + R\right)$
$Q_{i+1} = h^l - j < h^l - \frac{1}{2h^q}(h^{q-2} + 2h^{q+l}) = -\frac{1}{2h^2} < 0$
Hence,
$Q_{i+2} = h(h^l - j)^2 = \frac{1}{4h^{2q-1}}\left(K(q-2,0) + R\right)^2 > \frac{1}{2h^{2q-1}} R^2 > \frac{1}{2h^{2q-1}} 4h^{2q+l-2} = 2h^{l-1} > v$
Thus,$Q_{i+3} = \frac{1}{4h^{2q-2}}\left(K(q-2,0) + R\right)^2 - j = \frac{1}{4h^{2q-2}}\left(K(q-2,0) + R\right)^2 - \frac{1}{2h^q}\left(K(q-2,0) + 2h^{q+l} + R\right)$

$Q_{i+3} = \frac{1}{4h^{2q-2}}[(K(q-2,0) + R)^2 - 2h^{q-2}\left(K(q-2,0) + 2h^{q+l} + R\right)]$
$Q_{i+3} = \frac{1}{4h^{2q-2}}[R^2 + 2RK(q-2,0) + K(q-2,0)^2 - 2h^{q-2}K(q-2,0) - 4h^{2q+l-2} - 2h^{q-2}R]$

Due to (1) we obtain
$Q_{i+3} = \frac{1}{4h^{2q-2}}[R^2 + 2RK(q-3,0) + G(q) - 4h^{2q+l-2} - h^{2q-4}]$

$= \frac{1}{4h^{2q-2}}[R^2 + 2RK(q-3,0) + G(q) - 4h^{2q+l-2} - F(0,q)]$
$= \frac{1}{4h^{2q-2}}[R^2 + 2RK(q-3,0) + G(q) - 4K(0, 2q+l-2) - F(0,q)]$
Hence,
$Q_{i+3} > \frac{1}{4h^{2q-2}}[R^2 - 4K(0, 2q+l-2) - F(0,q)]$
$> \frac{1}{4h^{2q-2}}[G(q+1) + 4K(q-2, q+l) - 4K(0, 2q+l-2) - F(0,q)]$
$> \frac{1}{4h^{2q-2}}[F(0,q) + 4K(q-3, q+l) - F(0,q)]$
$> \frac{1}{h^{2q-2}} h^{2q+l-3} = h^{l-1} > v$
$Q_{i+4} = hQ_{i+3} - \frac{1}{2h^q}\left(K(q-2,0) + 2h^{q+l} + R\right)$
$= \frac{1}{4h^{2q-3}}[R^2 + 2RK(q-3,0) + G(q) - 4K(0, 2q+l-2) - F(0,q)] - \frac{1}{2h^q}\left(K(q-2,0) + 2h^{q+l} + R\right)$
$= \frac{1}{4h^{2q-3}}[R^2 + 2RK(q-3,0) + G(q) - 4K(0, 2q+l-2) - F(0,q) - 2h^{q-3}\left(K(q-2,0) + 2h^{q+l} + R\right)]$
$= \frac{1}{4h^{2q-3}}[R^2 + 2RK(q-4,0) + G(q) - 4K(0, 2q+l-2) - F(0,q) - 2h^{q-3}K(q-2,0) - 4h^{2q+l-3}]$
$= \frac{1}{4h^{2q-3}}[R^2 + 2RK(q-4,0) + G(q) - 4K(1, 2q+l-3) - F(0,q) - 2K(q-2, q-3)]$
According to lemma 3 ($u = 5$) we obtain
$Q_{i+4} = \frac{1}{4h^{2q-3}}[R^2 + 2RK(q-4,0) + G(q-1) - 4K(1, 2q+l-3) - F(1,q)]$
Further, we have
$Q_{i+4} > \frac{1}{4h^{2q-3}}[R^2 - 4K(1, 2q+l-2) - F(1,q)]$
$> \frac{1}{4h^{2q-3}}[G(q+1) + 4K(q-2, q+l) - 4K(1, 2q+l-3) - F(1,q)]$
$> \frac{1}{4h^{2q-3}}[4K(q-4, q+l)] = h^{l-1} > v$
Using induction we prove that the following statements are true

$Q_{i+u} = \frac{1}{4h^{2q-u+1}}[R^2 + 2RK(q-u,0) + G(q-u+3) - 4K(u-3, 2q+l-u+1) - F(u-3,q)]$,
$Q_{i+u} > h^{l-1} > v$ for $4 \leq u \leq q$

When we take $u = 4$ we have the basis step. It has been proved to be true in this case. Now assume that the relations are true for some u such that $4 \leq u \leq q$.

Since $Q_{i+u} > v$ we compute according to the definition

$Q_{i+u+1} = hQ_{i+u} - j =$

$\frac{1}{4h^{2q-u}}[R^2 + 2RK(q-u,0) + G(q-u+3) - 4K(u-3, 2q+l-u+1) - F(u-3,q)] - \frac{1}{2h^q}\left(K(q-2,0) + 2h^{q+l} + R\right) =$

$\frac{1}{4h^{2q-u}}[R^2 + 2RK(q-u,0) + G(q-u+3) - 4K(u-3, 2q+l-u+1) - F(u-3,q) - 2h^{q-u}(K(q-2,0) + 2h^{q+l} + R)] =$

$\frac{1}{4h^{2q-u}}[R^2 + 2RK(q-u-1,0) + G(q-u+3) - 4K(u-3, 2q+l-u+1) - F(u-3,q) - 2K(q-2,q-u) - 4h^{2q+l-u}] =$

$\frac{1}{4h^{2q-u}}[R^2 + 2RK(q-u-1,0) + G(q-u+3) - 4K(u-2, 2q+l-u) - F(u-3,q) - 2K(q-2,q-u)]$

According to lemma 3 we obtain

$$Q_{i+u+1} = \frac{1}{4h^{2q-u}}[R^2 + 2RK(q-u-1,0) + G(q-u+2) - 4K(u-2, 2q+l-u) - F(u-2,q)]$$

Now, we have

$$Q_{i+u+1} > \frac{1}{4h^{2q-u}}[R^2 - 4K(u-2, 2q+l-u) - F(u-2,q)]$$

$$> \frac{1}{4h^{2q-u}}[G(q+1) + 4K(q-2,q+l) - 4K(u-2, 2q+l-u) - F(u-2,q)]$$

$$> \frac{1}{4h^{2q-u}}[G(q+1) + 4K(q-2,q+l) - 4K(u-2, 2q+l-u) - F(u-2,q)]$$

$$> \frac{1}{4h^{2q-u}}[G(q+1) + 4K(q-u-1,q+l) - F(u-2,q)]$$

According to lemma 2 we obtain

$$Q_{i+u+1} > \frac{1}{4h^{2q-u}}[4K(q-u-1,q+l)] > h^{l-1} > v$$

We know now that $Q_{i+q} > v$. Hence, $Q_{i+q+1} = hQ_{i+q} - j$. Further,

$$Q_{i+q+1} = h(h(h...(h^2(h^l - j)^2 - j)...j) - j) - j \quad ... \ (3)$$

In (3) the number of j's is q. We rewrite (3) in the following equivalent form

$$Q_{i+q+1} = h^q j^2 - \left(2h^{q+l} + h^{q-2} + ... + h + 1\right) j + h^{q+2l}$$

The solution of the following equation

$$h^q x^2 - \left(2h^{q+l} + h^{q-2} + ... + h + 1\right) x + h^{q+2l} = h^m$$

or $h^q x^2 - \left(2h^{q+l} + K(q-2,0)\right) x + h^{q+2l} = h^m$ is

$x = \frac{1}{2h^q}\left(K(q-2,0) + 2h^{q+l} \pm \sqrt{(K(q-2,0) + 2h^{q+l})^2 + 4h^q(h^m - h^{q+2l})}\right)$

or

$x = \frac{1}{2h^q}\left(K(q-2,0) + 2h^{q+l} \pm \sqrt{4h^q h^m + K(q-2,0)^2 + 4h^{q+l}K(q-2,0)}\right)$

According to (2) we obtain

$x = \frac{1}{2h^q}(K(q-2,0) + 2h^{q+l} \pm \sqrt{4h^q h^m + G(q+1) + 4K(q-2,q+l)})$

But, our choice for j coincides with one of the roots. Hence, we shall have $Q_{i+q+1} = h^m$.

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