Theory of Multidimensional Differential Equations in Banach Space and Conditions of Full Solvability

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Abstract
In this paper it is the first time that the theorem of complete solvability for non-homogeneous multidimensional differential equation with special operator coefficients in Banach space is established. On proving this theorem it was suddenly discovered that \( n \)-dimensional vector space \( E^n \) (\( \dim E^n = n \)) turns into Lie algebra. At that there were obtained new conditions for complete solvability of multidimensional differential equations.

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1 Introduction
Let \( E^n \) be \( n \)-dimensional vector space, and \( E_y \) be arbitrary complex Banach space. Let the space of linear operators acting from \( E^n \) into \( E_y \) be designated as \( L(E^n;E_y) \). Let the space of strongly continuous mappings \( A : E^n \to L(E_y;E_y) \) of space \( E^n \) into Banach algebra \( L(E_x;E_y) \) of endomorphisms of Banach space \( E_y \) be designated as \( T(E_x;L(E_y;E_y)) \).

Let’s consider homogeneous multidimensional differential equation in the form:

\[
\begin{align*}
    u'(x)h &= A(Q(x)h)u(x) + \varphi h, \\
    u(x_0) &= u_0(x_0 \in S, u_0 \in E_y),
\end{align*}
\]

where derivative \( u'(x) \in L(E^n_x;E_y) \) — is understood in the sense of Fresher, \( x \in S, h \in E^n_x, u(x) \in E_y, Q(x) \in GL(n, R), S \) is a neighborhood of zero in \( E^n_x \). It is suggested that \( Q \in C^\infty(S) \) and \( Q(0) = I \) is an identity operator, \( A \in T(E^n_x;L(E_y;E_y)) \), at that \( A(h) \neq 0 \) on \( h \neq 0 \) and \( \varphi : E^n_x \to L(E^n_x;E_y) \) is a constant operator function.
2 The Basic Concepts

Definition 2.1 Equation (1) is called completely solvable (completely integrable), if for any point \((x_0, u_0), x_0 \in S, u_0 \in E_y\) there exists continuously differentiable solution \(u(x)\) of equation (1) defined in the neighborhood of \(x_0\) such that \(u(x_0) = u_0\).

Definition 2.2 Following Shilov [1], the mapping \(\omega : E_x \rightarrow E_y\) will be called morphism if there hold the following relations: 1) \(\omega(x_1 + x_2) = \omega(x_1) + \omega(x_2)\) for any \(x_1, x_2 \in E^n_x\); 2) \(\omega(\alpha x) = \alpha \omega(x)\) for any \(x \in E^n_x\) and any \(\alpha \in C\). If morphism \(\omega\) maps space \(E^n_x\) on to the whole space \(E_y\), it is called epimorphism. If morphism \(\omega\) maps space \(E^n_x\) at least not onto the whole space \(E_y\), but one-to-one (so that, from \(x_1 \neq x_2\) follows that \(\omega(x_1) \neq \omega(x_2)\)), it is called monomorphism.

Definition 2.3 Following N. Bourbaki [2] algebra \(\beta\) on \(K\) (\(K\) is commutative ring with a unit) is called Lie algebra on \(K\), if multiplication in it (designated by \((x, y) \rightarrow [x, y]\)) satisfies the identities

\[ a) \ [x, x] = 0, \ b) \ [x, y] = -[y, x], \ c) [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \]

where \(x, y, z\) are elements from \(\beta\) algebra. Product \([x, y]\) is called a commutator (sometimes - a bracket) "\(x\)" and "\(y\)". The identity (c) is called Jacoby's identity. It is clear that \([x, y]\) commutator is bilinear alternative function of elements "\(x\)" and "\(y\)", i.e. \(H(x, y) = [x, y] = -[y, x] = -H(y, x)\). \(\forall x, y \in \beta\).

3 Main Results

Theorem 3.1 Multidimensional differential equation (1) is completely solvable if and only if it is possible to represent Lie multiplication in the form

\[ \gamma(h, k) = [h, k] = Q'(x)(h, k) - Q'(x)(k, h) \quad \forall h, k \in E^n_x \]

so that there hold the following conditions

\[ A[h, k] = [Ah, Ak], \forall h, k \in E^n_x, \]

\[ Q'(x)(h, k) - Q'(x)(k, h) = [Q(x)h, Q(x)k], \]

\[ A(Q(x)h) \varphi k = A(Q(x)k) \varphi h. \]

Proof. Let us assume that equation (1) is completely solvable. This by definition means that for any point \((x_0, u_0), x_0 \in S, u_0 \in E_y\) there exists continuously differentiable solution of equation (1), defined in the neighborhood of the point such that \(u(x_0) = u_0\). Let function \(u(x)\) have such a solution. Substituting this solution into equation (1) we'll obtain the identity:

\[ u'(x)h \equiv A(Q(x)h)u(x) + \varphi h \quad (3) \]
Differentiating identity (3), we have:

\[ u''(x) (h, k) \equiv (u'(x) k)' h = A(Q'(x) (h, k))u(x) + A(Q(x) h) u'(x) k + 0 \]
\[ u''(x) (h, k) = A(Q'(x) (h, k))u(x) + A(Q(x) h) (A(Q(x) k) u(x) + \varphi k) = \]
\[ = (A(Q'(x) (h, k))u(x) + A(Q(x) h) A(Q(x) k) u(x) + A(Q(x) h) \varphi k = \]
\[ = (A(Q'(x) (k, h))u(x) + A(Q(x) k) A(Q(x) h) u(x) + A(Q(x) k) \varphi h. \]

(4)

By virtue of symmetry of bilinear mapping \( u''(x) \in L(E_x; L(E_x; E_y)) \), from (4) we obtain:

\[ A([Q'(x) (h, k) - Q'(x) (k, h)]u(x)) + [A(Q(x) h) A(Q(x) k) - \]
\[ - A(Q(x) k) A(Q(x) h) u(x) + AQ(x) h) \varphi k - A(Q(x) k) \varphi h = 0 \]

(5)

In relation (5) let us assume \( x = x_0 \) and notice, that \( u(x_0) = u_0 \) can be any vector from space \( E_y \).

At that from (5) for any \( x_0 \in S \) we find:

\[ A(Q'(x_0) (h, k) - Q'(x_0) (k, h)) = [A(Q(x_0) h) , A(Q(x_0) k)], \]

(6)

\[ A(Q(x_0) h) \varphi k - A(Q(x) k) \varphi h = 0, \]

(7)

where the square bracket means a commutator of operators:

\[ [A(Q(x_0) h) , A(Q(x_0) k)] = A(Q(x_0) h) A(Q(x_0) k) - \]
\[ - A(Q(x_0) k) A(Q(x_0) h) \]
\[ i.e. \ A(Q(x_0) h) , A(Q(x_0) k) \in L(E_y; E_y). \]

In equality (6) we assume \( x_0 = 0 \in S \), then we obtain:

\[ A(Q'(0) (h, k) - Q'(0) (k, h)) = [Ah, Ak], \forall h, k \in E^n_x \]

(8)

\[ A(Q(x_0) h) \varphi k - A(Q(x) k) \varphi h = 0 \]

(9)

Let’s introduce the designation:

\[ \gamma(h, k) = [h, k] = Q'(0) (h, k) - Q'(0) (k, h) \]

(10)

From relation (6) and monomorphism of mapping \( A \in L(E^n_x; L(E_y; E_y)) \) it follows that \( \gamma(h, k) \) specifies Lie multiplication in space \( E^n_x \) (see [3],[4]).

In this way it is not difficult to check that function \( \gamma(h, k) \) satisfies all the axioms a), b), c).

Thereby \( n \)-dimensional vector space \( E^n_x \) turns into Lie algebra. If designation (9) is taken into account in (8), then (8) takes the form:

\[ A [h, k] = [Ah, Ak] = AhAk - AkAh. \]

(11)
It is clear that operator $A \in L(E^n_x; L(E_y; E_y))$ is a morphism of Lie algebra into algebra $L(E_y; E_y)$ of endomorphisms of Banach space $E_y$.

Let us remind that operator $A : E^n_x \to L(E_y; E_y)$ is simultaneously a monomorphism. Now we’ll use this property of operator. At that from equality (6) we’ll obtain a commutative differential equation in the form:

$$Q'(x)(h, k) - Q'(x)(k, h) = [Q(x)h, Q(x)k](x \in S, h, k \in E^n_x).$$  \hspace{1cm} (12)$$

So for the complete integrability of equation (1) there is necessary the execution of the conditions:

$$A[h, k] = [Ah, Ak] = AhAk - AkAh;$$

$$A(Q(x)h)\varphi(k) = A(Q(x)k)\varphi(h);$$

$$Q'(x)(h, k) - Q'(x)(k, h) = [Q(x)h, Q(x)k]$$  \hspace{1cm} (13)

Applying G. Frobenius theorems (see [5]) on the complete solvability of multidimensional equations, we’ll obtain that conditions (13) are sufficient as well.

**Corollary 3.2** If $Q(x) = I$, $\forall x \in E^n_x$, then the equation takes the form:

$$u'(x)h = Ahu(x) + \varphi h,$$  \hspace{1cm} (14)

$$u(x_0) = u_0 \ (x \in S, u_0 \in E_y)$$  \hspace{1cm} (15)

Problem (14)-(15) is completely solvable if and only if there hold the conditions: $\Lambda_{hk}\{Ah, Ak\} = 0$, $\forall h, k \in E^n_x$ $A(h)\varphi(k) = A(k)\varphi(h)$, $\forall h, k \in E^n_x$, where $\Lambda_{hk}$ is an alternation operator, and has the following sense: $\Lambda_{hk}\{Bhk\} = \frac{1}{2}(Bhk - Bkh)$ for any bilinear operator $B : E^n_x \oplus E^n_x \to L(E_y; E_y)$. Let us consider functional-operator equation of the form (see [6])

$$\left(\lambda hI_{E_y} - Ah\right)y = \varphi h, \ (h \in E^n_x)$$  \hspace{1cm} (16)

where is $\varphi : E^n_x \to L(E^n_x; E_y)$ a known operator, $I$ is the identity operator from algebra $L(E_y; E_y)$, $\lambda : E^n_x \to C$ is a linear functional, $A \in L(E^n_x; L(E_y; E_y))$ is an operator, satisfying the condition: $\Lambda_{hk}\{AhAk\} = 0 \ \forall h, k \in E^n_x$ where $\Lambda_{hk}$ designates an alternation operator, $y \in E_y$ is an unknown one.

**Corollary 3.3** Let $0 \neq \lambda \in \sigma(A) \subset E^*_x$. Then operator equation (16) is completely solvable if and only if there holds the condition:

$$\left(\lambda hI_{E_y} - Ah\right)\varphi(k) = \left(\lambda kI_{E_y} - A(k)\right)\varphi(h) \ \forall h, k \in E^n_x.$$

Here $\sigma(A)$ is a spectrum of operator $A \in L(E^n_x; L(E_y; E_y))$, and $E^*_x$ is conjugated to space $E_x$. 
Multidimensional differential equations in Banach space

References


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