Spectral Theory and Functional Calculus of a Pair of Operators Acting from One Complex Banach Space to Another

M. B. Ragimov

Baku State University, Az-1148, Z.Khalilov 23, Baku, Azerbaijan
rahimovmisir@yahoo.com

Y. Y. Mustafayeva

Baku State University, Az-1148, Z.Khalilov 23, Baku, Azerbaijan
helenmust@rambler.ru

Abstract

In the paper spectral theory and functional calculus of a pair of perturbed linear operators acting from one complex Banach space $X$ to another $Y$ is built. As distinct from the spectrum of a bounded operator acting in one space in this case the spectrum of a pair of operators $(A, B)$ from $X$ to $Y$ can be for instance an unbounded set. Besides that the spectrum $\sigma(A, B)$ can be an empty set.

Mathematics Subject Classification: primary 47G99, secondary 47A99

Keywords: pair of operators, Banach space, spectrum, perturbation, left (right) pseudoresolvent

1 Introduction

Let’s consider ordered pairs of linear operators $(A, B)$ from Banach space $L(X, Y)$ of linear bounded operators defined on a complex Banach space $X$ with values in another complex Banach space $Y$ [1].

2 Preliminary Notes

Definition 2.1 The set of such complex numbers $\lambda \in \mathbb{C}$ for which operator $A - \lambda B$ where $A, B \in L(X, Y)$ doesn’t have the bounded inverse one is called
the spectrum of the pair of operators \((A, B)\). Let’s designate the spectrum of
of the pair \((A, B)\) as \(\sigma(A, B)\). The regular set of the pair \(A, B \in L(X, Y)\)
is designated as \(\rho(A, B)\), i.e. \(\rho(A, B) = C \setminus \sigma(A, B)\).

**Definition 2.2** Function \(R_{AB}(\cdot) = R(\cdot, A, B) : \rho(A, B) \to L(Y, X)\) defined
by the equality

\[
R_{AB}(\cdot) = (A - \lambda B)^{-1}, \ \lambda \in \rho(A, B),
\]

is called the resolvent of the pair of operators \((A, B)\) from \(L(X, Y)\).

**Definition 2.3** Extended spectrum \(\tilde{\sigma}(A, B)\) is a subset of extended complex
plane \(\bar{C} = C \cup \{\infty\}\) which coincides with \(\sigma(A, B)\) if both functions
\(\lambda \mapsto B(A - \lambda B)^{-1} : \bar{C} \to L(Y)\) and \(\lambda \mapsto (A - \lambda B)^{-1}B : \bar{C} \to L(X)\) are holomorphic at
the point \(\infty\) and coincide with \(\sigma(A, B) \cup \{\infty\}\) otherwise. Suppose
\(\tilde{\rho}(A, B) = \bar{C} \setminus \tilde{\sigma}(A, B)\).

**Definition 2.4** Functions \(R_l = BR(\lambda; A, B) : \rho(A, B) \to L(Y)\), \(R_r = R(\lambda; A, B)B : \rho(A, B) \to L(X)\) are called left and right pseudoresolvents
respectively.

### 3 Main Results

These are the main results of the paper.

**Theorem 3.1** If \(A, B \in L(X, Y)\) and \(\lambda, \mu \in \rho(A, B)\) then holds the equality

\[
R(\lambda, A, B) - R(\mu, A, B) = -(\mu - \lambda)R(\lambda, A, B)BR(\mu, A, B).
\]

**Proof.** Consider the equality:

\[
A - \lambda B)[(A - \lambda B)^{-1} - (A - \mu B)^{-1}](A - \mu B) =
\]

\[
= [I_Y - (A - \lambda B)(A - \mu B)^{-1}](A - \mu B) =
\]

\[
= [(A - \mu B) - (A - \lambda B)] = -(\mu - \lambda)B.
\]

Thus we obtain

\[
(A - \lambda B)[(A - \lambda B)^{-1} - (A - \mu B)^{-1}](A - \mu B) = -(\mu - \lambda)B.
\]

Hence multiplying (3) from the left and from the right by operators \((A - \lambda B)^{-1}\)
and \((A - \mu B)^{-1}\) respectively we get from (2):

\[
[(A - \lambda B)^{-1} - (A - \mu B)^{-1}] = -(\mu - \lambda)(A - \lambda B)^{-1}B (A - \mu B)^{-1}.
\]

**Corollary 3.2** Acting on equality (3) from the left and the right by operator
\(B : X \to Y\) we obtain:

\[
R_l(\lambda) - R_l(\mu) = -(\mu - \lambda)R_l(\lambda)R_l(\mu),
\]

\[
R_r(\lambda) - R_r(\mu) = -(\mu - \lambda)R_r(\lambda)R_r(\mu),
\]

where \(R_l(\lambda)\) and \(R_r(\lambda)\) are the left and the right pseudoresolvents.
Theorem 3.3 Let $A, B$ be elements from $L(X,Y)$. Let $\lambda, \mu \in \rho(A,B)$. Then operators $R_{AB}(\lambda)$, $R_{AB}(\mu) : Y \rightarrow X$ commute.

Proof. As $\lambda, \mu \in \rho(A,B)$ theorem 3.1 is valid i.e.

$$R(\lambda, A, B) \cdot R(\mu, A, B) = -(\mu - \lambda) R(\lambda, A, B) BR(\mu, A, B),$$

$$R(\mu, A, B) - R(\lambda, A, B) = -(\lambda - \mu) R(\mu, A, B) BR(\lambda, A, B),$$

what proves the theorem.

Let $\mu \rightarrow T(\mu)$ as an operator-valued function of complex parameter $\mu$ be continuous (and analytical) in uniform topology. Our purpose is to investigate how the spectrum and the resolvent of the pair $(T(\mu), B)$ change at small changes of $\mu$. The main result here is this theorem describing how analytical points of the spectrum of $(T(0), B)$ change if $(T(\mu), B)$ analytically depends on $\mu$.

Let’s designate regular sets of pairs of operators $(T(\mu), B)$ and $(T(0), B) \in L(X,Y)$ as $\rho(T(0), B)$ and $\rho(T(\mu), B)$ respectively.

Theorem 3.4 Let $(T(\mu), B) \in L(X,Y)$ at that $T(\mu)$ be analytical in the circle $|\mu| < \gamma$ ($\gamma > 0$), besides that $\lambda \in \rho(T(0), B) \cap \rho(T(\mu), B)$ and holds the condition $\|T(\mu) - T(0)\| < \inf \|R(\lambda; T(0), B)\|^{-1}, \lambda \in U$. Suppose that $U$ is such an open set that $\overline{U} \subset \rho(T(0), B)$. Then $\exists$ such a number $\delta > 0$ that at $|\mu| < \delta$ holds the relationship: $\overline{U} \subset \rho(T(\mu), B)$ and the resolvent $R(\lambda, T(\mu), B) = (T(\mu) - \lambda B)^{-1} : \rho(T(\mu), B) \rightarrow L(Y, X)$ of the pair of operators $(T(\mu), B) : X \rightarrow Y$ is analytical function with respect to $\mu$ in the circle $|\mu| < \delta$ for all $\lambda \in U$.

Proof. Let $\delta_1 > 0$ be chosen so that $\delta < \delta_1, |\mu| < \delta < \delta_1$ and $\overline{U} \subset \rho(T(\mu), B)$. At that holds the condition of the theorem: $\|T(\mu) - T(0)\| < \inf \|R(\lambda; T(0), B)\|^{-1}, \lambda \in U$.

As in theorem 3.1 consider the equality:

$$(T(\mu) - \lambda B) [(T(\mu) - \lambda B)^{-1} - (T(0) - \lambda B)^{-1}](T(0) - \lambda B) =
= [(I_T - (T(\mu) - \lambda B)(T(0) - \lambda B)^{-1}][T(0) - \lambda B] = [(T(0) - \lambda B) - (T(\mu) - \lambda B)] = T(0) - T(\mu).$$

Thus we have

$$(T(\mu) - \lambda B)[(T(\mu) - \lambda B)^{-1} - (T(0) - \lambda B)^{-1}][T(0) - \lambda B] = T(0) - T(\mu). \quad (7)$$

It is clear that from relationship (7) it follows:

$$(T(\mu) - \lambda B)^{-1} - (T(0) - \lambda B)^{-1} = (T(\mu) - \lambda B)^{-1}(T(0) - T(\mu))(T(0) - \lambda B)^{-1}. \quad (8)$$
From equality (10) we finally have:

\[(T(\mu) - \lambda B)^{-1}[I_Y - (T(0) - T(\mu))(T(0) - \lambda B)^{-1}] = (T(0) - \lambda B)^{-1}. \tag{9}\]

By condition \[\|I_Y - (T(0) - T(\mu))(T(0) - \lambda B)^{-1}\| < 1.\] At that operator \[I_Y - (T(0) - T(\mu))(T(0) - \lambda B)^{-1}: Y \to Y\] has the bounded inverse one.

Multiplying operator relationship (9) by the inverse operator \[I_Y - (T(0) - T(\mu))(T(0) - \lambda B)^{-1}]^{-1} \in L(Y)\] from the right we have

\[(T(\mu) - \lambda B)^{-1} = (T(0) - \lambda B)^{-1}[I_Y - (T(0) - T(\mu))(T(0) - \lambda B)^{-1}]^{-1}. \tag{10}\]

From equality (10) we finally have:

\[R(\lambda; T(\mu), B) = (T(\mu) - \lambda B)^{-1} = (T(0) - \lambda B)^{-1} \sum_{n=0}^{\infty} [(T(0) - T(\mu))(T(0) - \lambda B)^{-1}]^n.\]

At \(|\mu| < \delta\) this series converges absolutely and uniformly. So the resolvent \[R(\lambda; T(\mu), B) = (T(\mu) - \lambda B)^{-1}\] of the pair of operators \((T(\mu), B) \in L(X, Y)\) is analytical in the circle \(|\mu| < \delta\) at all \(\lambda \in U\). The theorem is proved.

Let’s designate the set of analytical functions in the neighborhood of spectrum \(\sigma(A, B)\) of the pair of operators \((A, B)\) as \(F(\sigma(A, B))\).

**Theorem 3.5** Let \((A, B) \in L(X, Y)\), at that \(\|AB^{-1}\| < 1\), and function \(f \in F(\sigma(A, B))\). Then \(T(f) = -\sum_{n=0}^{\infty} \alpha_n B^{-1}(AB^{-1})^n\) and the series converges in uniform operator topology.

**Proof.** Consider the bundle \((A - \lambda B) = (AB^{-1} - \lambda I_Y)B\) and \(-(\lambda B - A) = -(\lambda I_Y - AB^{-1})B\) Hence at \(\lambda \in \rho(A, B)\) we have:

\[-(\lambda B - A)^{-1} = -B^{-1}(\lambda I_Y - AB^{-1})^{-1} \in L(Y, X).\]

As function \(f \in F(\sigma(A, B))\) then in the neighborhood of spectrum \(\sigma(A, B)\) it is discomposed in the convergent series \(f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n\).

Now the proof of the theorem is obtained from the equalities:

\[T(f) = \frac{1}{2\pi i} \int_{\gamma} f(\lambda)(A - \lambda B)^{-1}d\lambda = -\frac{1}{2\pi i} \sum_{n=0}^{\infty} \alpha_n \int_{\gamma} \lambda^n B^{-1}(\lambda I_Y - AB^{-1})^{-1}d\lambda = -\sum_{n=0}^{\infty} \alpha_n B^{-1} \left\{ \frac{1}{2\pi i} \int_{\gamma} \lambda^n B^{-1}(\lambda I_Y - AB^{-1})^{-1}d\lambda \right\} = -\sum_{n=0}^{\infty} \alpha_n B^{-1}(AB^{-1})^n.\]

Thus \(T(f) = -\sum_{n=0}^{\infty} \alpha_n B^{-1}(AB^{-1})^n \in L(Y; X)\).

The theorem is proved.

Let’s consider a generalization of theorem 6.10 [2, chap.VII, p.630] for a pair of operators \((A, B)\) from \(L(X, Y)\). For this purpose it is convenient to define the resolvent \(R(\lambda; A, B)\) of the pair \((A, B)\) in the form of \((\lambda B - A)^{-1}\). Then similar to Danford’s operator-function we’ll define an operator-function \(f(A, B)\) of a pair of operators \((A, B)\) where function \(f(\lambda) \in F(\sigma(A, B))\) as follows: \(f(A, B) = \int f(\lambda)R(\lambda; A, B)d\lambda\).
**Theorem 3.6** Let operators \( A, B, N \in L(X, Y) \) and operator \( B \) have bounded inverse \( B^{-1} \). Let function \( f(\lambda) \in F(\sigma(A, B)) \). Suppose spectrum \( \sigma(N, B) \) of the pair of operators \((N, B)\) lies inside an open circle with radius \( \varepsilon \) with the center at the origin of the system. Then function \( f(\lambda) \) is analytical in some neighborhood of the spectrum \( \sigma(A + N, B) \) of the pair of operators \( A + N, B : \)

\[
f(A + N, B) = \sum_{n=0}^{\infty} \frac{f^{(n)}(A, B)(NB^{-1})^n}{n!},
\]

at that the series is convergent in uniform operator topology.

**Lemma 3.7** Let \( S \) be a set the distance from which to the spectrum \( \sigma(G, B) \) of the pair \( (G, B) \in L(X, Y) \) is greater than some positive number \( \varepsilon \) and let operator \( B \) have the bounded inverse operator \( B^{-1} \). Then there exists such a constant \( K \) that

\[
|R(\lambda; G, B)| < K \cdot \varepsilon^{-n}, \quad n \geq 0, \quad \lambda \in S.
\]

**Proof.** Let an open set \( U \supset \sigma(G, B) \) and have boundary \( \Gamma \) consisting of a finite number of rectifiable Jordan curves. Suppose that for all \( \lambda \in \sigma(G, B) \) and any \( \alpha \in U \cup \Gamma \) we have \( |\lambda - \alpha| > \varepsilon \). Then as resolvent \( R(\alpha, B^{-1}G) \) is bounded on \( \Gamma \) we get the following:

\[
|\{R(\lambda; G, B)B^n\}| = \left|\left[(G - \lambda B)^{-1}B\right]^n\right| = \left|\left[(B^{-1}G - \lambda I(X))^{-1}\right]^n\right| =
\]

\[
= \frac{1}{2\pi i} \int_{\Gamma} (\alpha - \lambda)^{-n} R(\alpha, B^{-1}G) d\alpha \leq K \cdot \varepsilon^{-n}.
\]

**Proof of theorem 3.6.** Let \( \delta = \sup |\lambda|, \lambda \in \sigma(N, B) \). Choose \( \theta < 1 \) so that \( \delta < \theta \varepsilon < \varepsilon \) and let \( M \) be a circumference (\( \lambda : |\lambda| = \theta \varepsilon \)). Then

\[
|(B^{-1}N)^n| = \left|\frac{1}{2\pi i} \int_{M} \lambda^n R(\lambda; B^{-1}N) d\lambda\right| \leq K(\theta \varepsilon)^{n+1}.
\]

This inequality with the lemma shows that the series

\[
V = \sum_{n=0}^{\infty} [R(\lambda; A, B)]^{n+1}(B^{-1}N)^n : X \rightarrow X
\]

converges uniformly by \( \lambda \) from \( C : \rho(C, \sigma(N, B)) < \varepsilon \).

Evidently, \( VB^{-1}(\lambda B - (A + N)) = I_X \).

Indeed, as

\[
[R(\lambda; A + N, B)B]^{-1} = B^{-1}(\lambda B - (A + N)) = [R(\lambda; A, B)B]^{-1} - B^{-1}N
\]

we have

\[
VB^{-1}(\lambda B - (A + N)) = V[R(\lambda; A + N, B)B]^{-1} =
\]

\[
= \sum_{n=0}^{\infty} [R(\lambda; A, B)B]^{n}(B^{-1}N)^n - \sum_{n=1}^{\infty} [R(\lambda; A, B)B]^{n}(B^{-1}N)^n = I_X.
\]

Consequently,

\[
(\lambda B - A - N)^{-1}B = V = \sum_{n=0}^{\infty} [R(\lambda; A, B)B]^{n+1}(B^{-1}N)^n,
\]

or

\[
R(\lambda; A + N, B)B = \sum_{n=0}^{\infty} [R(\lambda; A, B)B]^{n+1}(B^{-1}N)^n.
\]

So function \( f \) from theorem 3.6 is analytical in the neighborhood of the spectrum \( \sigma(A + N, B) \).
Let $\Gamma$ designate a union of finite collection $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ of nonintersecting closed rectifiable Jordan contours which bound the domain $D$ containing $\varepsilon-$neighborhood of spectrum $\sigma(A, B)$ and lie together with domain $D$ totally in the domain of analyticity of function $f$. Suppose that $\Gamma_i, i = 1, 2, \ldots, m,$ are positively oriented. Then

$$ f(A + N, B) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R(\lambda; A + N, B) d\lambda = $$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{\Gamma} f(\lambda) [R(\lambda; A, B) B]^{n+1} d\lambda (B^{-1} N)^n B^{-1}. $$

On the other hand, from Hilbert identity it follows that $R(\lambda_1; A, B) - R(\lambda_2; A, B) = (\lambda_2 - \lambda_1) R(\lambda_1; A, B) B R(\lambda_2; A, B)$. Hence

$$ \frac{d}{d\lambda} R(\lambda; A, B) = -R(\lambda; A, B) B R(\lambda; A, B). $$

By induction

$$ \left(\frac{d}{d\lambda}\right)^n R(\lambda; A, B) = (-1)^n n! [R(\lambda; A, B) B]^{n+1} B^{-1}. $$

Hence

$$ \int_{\Gamma} f(\lambda) [R(\lambda; A, B) B]^{n+1} d\lambda = \frac{(-1)^n}{n!} \int_{\Gamma} f(\lambda) \left(\frac{d}{d\lambda}\right)^n R(\lambda; A, B) B d\lambda. $$

Integrating by parts we find

$$ \int_{\Gamma} f(\lambda) [R(\lambda; A, B) B]^{n+1} d\lambda = \frac{1}{n!} \int_{\Gamma} \left\{ \left(\frac{d}{d\lambda}\right)^n f(\lambda) \right\} R(\lambda; A, B) B d\lambda, $$

whence

$$ f(A + N, B) = \sum_{n=0}^{\infty} \left\{ \frac{1}{2\pi i} \int_{\Gamma} f^{(n)}(\lambda) R(\lambda; A, B) B d\lambda \right\} (B^{-1} N)^n B^{-1} = $$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(A, B) B [B^{-1} N]^n B^{-1}}{n!} = \sum_{n=0}^{\infty} \frac{f^{(n)}(A, B) [NB^{-1}]^n}{n!}, $$

which is convergent in the uniform operator topology, what was required to prove.

References


Received: September 18, 2008