Some New Inequalities of Hardy-Hilbert’s Type

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Abstract. In this paper, some new inequalities with \((p,q)\)-parameters are obtained for both discrete and integral forms. As an application we give the equivalent form in each case.

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1. Introduction

If \(p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x), g(x) > 0, 0 < \int_0^\infty f^p(x)dx < \infty, \text{and } 0 < \int_0^\infty g^q(x)dx < \infty\), then we have the following inequality

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} \,dx\,dy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(x)dx \right\}^{\frac{1}{q}}
\]

where the constant factor \(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\) is the best possible. The double series inequality is

\[
\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \sum_{n=0}^\infty a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^\infty b_n^q \right\}^{\frac{1}{q}},
\]

where the sequences \(\{a_n\}\) and \(\{b_n\}\) are such that \(0 < \sum_{n=0}^\infty a_n^p < \infty, 0 < \sum_{n=0}^\infty a_n^q < \infty\), and the constant factor \(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\) is also the best possible. Inequalities (1.1) and (1.2) are called Hardy-Hilbert’s inequalities(see [1]), which plays an important role in analysis and its applications (see [2]). In the last decade many generalizations of both (1.1) and (1.2) were given, for an example (see [4,5]).
Under the same conditions in (1) and (2) respectively, Hardy [1] gave the following inequalities

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}} \, dx \, dy < pq \left\{ \int_0^\infty f^p(x) \, dx \right\}^{\frac{1}{q}} \left\{ \int_0^\infty g^q(x) \, dx \right\}^{\frac{1}{p}}, \tag{1.3}
\]

\[
\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{\max\{m,n\}} < pq \left\{ \sum_{n=1}^\infty a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty b_n^q \right\}^{\frac{1}{q}}, \tag{1.4}
\]

where the constant factor \(pq\) is the best possible in both (1.3) and (1.4). In particular, when \(p = q = 2\) we have the well-known Hilbert type inequality:

\[
\sum_{n=1}^\infty \sum_{m=1}^\infty |\ln m/n| a_m a_n \leq J \sum_{n=1}^\infty \sum_{m=1}^\infty K_1(m,n) a_m a_n, \tag{1.5}
\]

where \(J = \int_0^\infty \frac{K_1(\omega,1)}{\sqrt{\omega}} \, d\omega\).

As a consequence of this result, Hardy gave the following inequality

\[
\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{\ln \frac{m}{n}}{\max\{m,n\}} a_m a_n \leq 2 \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m a_n}{\max\{m,n\}}. \tag{1.6}
\]

Our main goal in this paper is to obtain a new inequality related to the double series \(\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{\ln \frac{m}{n}}{\max\{m,n\}} a_m b_n\) with \((p,q)\)-parameters. To do this, we will give some new integral inequalities and use them to obtain our result.

2. Two Lemmas

Lemma 2.1. Suppose that \(r > 1, \frac{1}{r} + \frac{1}{s} = 1\), and define \(\omega(r,t)\) as

\[
\omega(r,t) = \int_0^1 \frac{1 + |\ln \frac{z}{t}|}{\max\{t,z\}} \left( \frac{t}{z} \right)^{\frac{1}{r}} \, dz.
\]

Then, we get
(2.1) \[ \omega(r,t) = s^2 + rs + r^2. \]

**Proof.** For a fixed \( t \), let \( u = \frac{t}{z} \), then

\[
\int_0^\infty \frac{1 + |\ln t/z|}{\max \{t,z\}} \left( \frac{t}{z} \right)^{\frac{1}{r}} \, dz = \int_0^\infty \frac{1 + |\ln u|}{\max \{1,u\}} \left( \frac{1}{u} \right)^{\frac{1}{r}} \, du
\]

\[
= \int_0^1 (1 - \ln u) u^{-\frac{1}{r}} \, du + \int_1^\infty (1 + \ln u) u^{-1-\frac{1}{r}} \, du.
\]

Using integration by parts, we get (2.1).

**Lemma 2.2.** Suppose that \( s > 1, \frac{1}{s} + \frac{1}{r} = 1 \), and \( 0 < \varepsilon < s - 1 \). Then

(2.2) \[ \int_1^\infty y^{-1-\varepsilon} \int_0^\frac{1}{y} \frac{1 + |\ln u|}{\max \{1,u\}} u^{-\frac{1}{s} + \frac{1}{r}} \, dy \, du = O(1) \quad (\varepsilon \to 0^+). \]

**Proof.** Since \( y \geq 1 \), then

\[
\int_0^{\frac{1}{y}} \frac{1 + |\ln u|}{\max \{1,u\}} u^{-\frac{1}{s} + \frac{1}{r}} \, du = \int_0^{\frac{1}{y}} (1 - \ln u) u^{-\frac{1}{s} + \frac{1}{r}} \, du
\]

\[
= \frac{rs}{s - \varepsilon r} \left( \ln y + \frac{rs}{s - \varepsilon r} + 1 \right) y^{\frac{r}{s} - \frac{1}{r}}.
\]

So, we obtain

\[
\int_1^\infty y^{-1-\varepsilon} \int_0^{\frac{1}{y}} \frac{1 + |\ln u|}{\max \{1,u\}} u^{-\frac{1}{s} + \frac{1}{r}} \, dy \, du
\]

\[
= \frac{rs}{s - \varepsilon r} \int_1^\infty y^{-\frac{s}{r} - \frac{1}{r} - 1} \left( \ln y + \frac{rs}{s - \varepsilon r} + 1 \right) \, dy
\]

\[
= \frac{rs}{s - \varepsilon r} \left( \left( \frac{r}{\varepsilon + 1} \right)^2 + \frac{r}{\varepsilon + 1} \left( \frac{rs}{s - \varepsilon r} + 1 \right) \right). \quad (\varepsilon \to 0^+)
\]
3. Main Results

Theorem 3.1. Suppose that \( f(x), g(x) \geq 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1, \) and \( 0 < \int_0 f^p(x)dx < \infty, < \int_0 g^q(x)dx < \infty. \) Then

\[
\int_0 \int_0 1 + \left| \frac{\ln x}{y} \right| f(x)g(y)dydx < (p^2 + pq + q^2) \left\{ \int_0 f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0 g^q(x)dx \right\}^{\frac{1}{q}},
\]

where the constant factor \((p^2 + pq + q^2)\) is the best possible. In particular, if \( p = q = 2 \), we get

\[
\int_0 \int_0 1 + \left| \frac{\ln x}{y} \right| f(x)g(y)dydx < 12 \left\{ \int_0 f^2(x)dx \right\}^{\frac{1}{2}} \left\{ \int_0 g^2(x)dx \right\}^{\frac{1}{2}}.
\]

Proof. Using Holder’s inequality, we obtain

\[
\int_0 \int_0 1 + \left| \frac{\ln x}{y} \right| f(x)g(y)dydx = \int_0 \int_0 \left\{ \frac{1 + \left| \frac{\ln x}{y} \right|}{\max \{x, y\}} \right\} \left[ x \frac{1}{y} g(y) \right] f(x) dx dy
\]

\[
\leq \left\{ \int_0 \int_0 \frac{1 + \left| \frac{\ln x}{y} \right|}{\max \{x, y\}} \left[ x \frac{1}{y} \right] f^p(x) dy dx \right\}^{\frac{1}{p}} \times \left\{ \int_0 \int_0 \frac{1 + \left| \frac{\ln x}{y} \right|}{\max \{x, y\}} \left[ y \frac{1}{x} \right] g^q(y) dy dx \right\}^{\frac{1}{q}}
\]

\[
(3.2)
\]

where \( \omega(q, x) = \int_0^{\infty} 1 + \left| \frac{\ln x}{y} \right| \left[ \frac{x}{y} \right]^{\frac{1}{q}} dy, \) and \( \omega(p, y) = \int_0^{\infty} 1 + \left| \frac{\ln x}{y} \right| \left[ \frac{y}{x} \right]^{\frac{1}{p}} dx. \) By (2.1) we have
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\( \omega(q, x) = \omega(p, y) = p^2 + pq + q^2. \)

If (3.2) takes the form of equality, then there exists real numbers \( A \) and \( B \) such that they are not all zero and,

\[
A \frac{1 + \left| \ln \frac{x}{y} \right|}{\max \{ x, y \}} \left[ x \right]^{\frac{1}{p}} f^p(x) = B \frac{1 + \left| \ln \frac{x}{y} \right|}{\max \{ x, y \}} \left[ y \right]^{\frac{1}{p}} g^q(y), \text{ a.e. in } (0, \infty) \times (0, \infty).
\]

Hence, we find

\[ Axf^p(x) = Byg^q(y), \text{ a.e. in } (0, \infty) \times (0, \infty). \]

It follows that there exists a constant \( C \), such that

\[ Axf^p(x) = C, \text{ a.e. in } (0, \infty) \]

\[ By^qg^q(y) = C \text{ a.e. in } (0, \infty). \]

Without loss of generality, suppose that \( A \neq 0 \). Then we have

\[ f^p(x) = \frac{C}{Ax}, \text{ a.e. in } (0, \infty), \]

which contradicts the fact that \( 0 < \int_0^\infty f^p(x) dx < \infty \). Therefore, (3.2) takes the form of strict inequality, and we may rewrite it as

\[
\int_0^\infty \int_0^\infty \frac{1 + \left| \ln \frac{x}{y} \right|}{\max \{ x, y \}} f(x)g(y)dxdy < \left\{ \int_0^\infty \omega(q, x) f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \omega(p, y) g^q(y) dy \right\}^{\frac{1}{q}}.
\]

From (3.3) and (3.4) we get (3.1).

We need to show that the constant factor \( p^2 + pq + q^2 \) contained in (3.1) is the best possible. To do that we define two functions

\[ \tilde{f}(x) = \begin{cases} 0, & x \in (0, 1) \\ x^{-\frac{1}{p}}, & x \in [1, +\infty) \end{cases} \]

and
\[
\tilde{g}(y) = \begin{cases} 
0, & y \in (0, 1) \\
y^{-\frac{1+\varepsilon}{q}}, & y \in [1, +\infty)
\end{cases}.
\]

Assume that \( p > 1 \), and \( 0 < \varepsilon < q - 1 \). Suppose that \( p^2 + pq + q^2 \) is not the best possible, then there exists \( 0 < K < p^2 + pq + q^2 \) such that

\[
\int_0^\infty \int_0^\infty \frac{1 + \left| \ln \frac{x}{y} \right|}{\max \{x, y\}} f((x) \tilde{g}((y) dxdy < K \left\{ \int_0^\infty f^p((x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \tilde{g}^q((y) dy \right\}^{\frac{1}{q}}
\]

(3.5)

\[
= K \left\{ \int_1^\infty x^{-1-\varepsilon} dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty y^{1-\varepsilon} dy \right\}^{\frac{1}{q}} = \frac{K}{\varepsilon}
\]

On the other hand, for a fixed \( y \) setting \( u = \frac{x}{y} \), by (2.2) we have

\[
\int_0^\infty \int_0^\infty \frac{1 + \left| \ln \frac{x}{y} \right|}{\max \{x, y\}} f((x) \tilde{g}((y) dxdy = \int_1^\infty \int_1^{y^{-1}} \frac{1 + \left| \ln u \right|}{\max \{u, 1\}} u^{-\frac{1+\varepsilon}{p}} du \left\{ \int_0^y f^p((x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \tilde{g}^q((y) dy \right\}^{\frac{1}{q}}
\]

(3.6)

\[
= \frac{p^2 + pq + q^2 + o(1)}{\varepsilon} - \int_1^{y^{-1-\varepsilon}} \int_0^{y^{-1}} \frac{1 + \left| \ln u \right|}{\max \{u, 1\}} u^{-\frac{1+\varepsilon}{p}} du dy
\]

\[
= \frac{p^2 + pq + q^2 + o(1)}{\varepsilon} - O(1)
\]

\[
= \frac{p^2 + pq + q^2 + o(1)}{\varepsilon} (\varepsilon \to 0^+)
\]

(3.6)

Clearly, when \( \varepsilon \to 0^+ \), the inequality (3.5) is in contradiction with (3.6). Thus the constant factor \( p^2 + pq + q^2 \) is the best possible. The proof of the theorem is completed.

**Corollary 3.2.** Under the same conditions in Theorem 3.1, we still have

(3.7)

\[
I_1 := \int_0^\infty \int_0^\infty \frac{1 + \left| \ln \frac{x}{y} \right|}{\max \{x, y\}} f(x)g(y) dxdy \leq (p^2 + q^2) \left\{ \int_0^\infty f^p((x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q((x) dx \right\}^{\frac{1}{q}}
\]
where the constant $p^2 + q^2$ factor is the best possible. In particular when $p = q = 2$, we get

$$\int_0^\infty \int_0^\infty \frac{\ln xy}{\max \{x, y\}} f(x)g(y)dxdy \leq 8 \left\{ \int_0^\infty f^2(x)dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x)dx \right\}^{\frac{1}{2}}.$$

**Proof.** Note that the double integral $I$ in Theorem 3.1 can be written as

$$I = \int_0^\infty \int_0^\infty \frac{1 + \left| \frac{\ln xy}{\max \{x, y\}} \right| f(x)g(y)dxdy = \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max \{x, y\}} dxdy + \int_0^\infty \int_0^\infty \left| \frac{\ln xy}{\max \{x, y\}} \right| f(x)g(y)dxdy.$$

Therefore, (3.1) can be written as

$$I = I_1 + I_2 < pqF^\frac{1}{p}G^\frac{1}{q} + (p^2 + q^2) F^\frac{1}{p}G^\frac{1}{q},$$

where $F^\frac{1}{p} = \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}}$ and $G^\frac{1}{q} = \left\{ \int_0^\infty g^q(x)dx \right\}^{\frac{1}{q}}$. Suppose that $I_2 > (p^2 + q^2) F^\frac{1}{p}G^\frac{1}{q}$, then we obtain

$$I_1 = I - I_2 < (p^2 + q^2) F^\frac{1}{p}G^\frac{1}{q} + pqF^\frac{1}{p}G^\frac{1}{q} - I_2 = M + pqF^\frac{1}{p}G^\frac{1}{q} < pqF^\frac{1}{p}G^\frac{1}{q}. \quad (M < 0)$$

which contradicts the fact that $pq$ is the best possible constant in inequality (1.3). Thus, inequality (3.7) is valid. If the constant factor $p^2 + q^2$ is not the best possible in (3.7), then we may get a contradiction that the constant factor in (3.1) is not the best possible.

**Remark 3.3.** Note that, we can get inequality (3.7) by the same method in Theorem 3.1 and the obtained inequality will be a strict one.

**Theorem 3.4.** If $f(x), g(x) \geq 0$, $p > 1$, such that $0 < \int_0^\infty f^p(x)dx < \infty$, $0 < \int_0^\infty g^q(y)dy < \infty$, then:

$$\int_0^\infty \left[ \int_0^\infty \frac{1 + \left| \frac{\ln xy}{\max \{x, y\}} \right| f(x)}{\max \{x, y\}} dx \right]^p dy < [p^2 + pq + q^2]^p \int_0^\infty f^p(x)dx,$$
and

\[(3.9) \quad \int_0^\infty \left[ \int_0^\infty \left| \frac{\ln x}{y} \right| f(x) dx \right] dy < \left[ p^2 + q^2 \right]^p \int_0^\infty f^p(x) dx, \]

where the constant factor \([p^2 + pq + q^2]^p\) and \([p^2 + q^2]^p\) are the best possible. Inequality (3.8) is equivalent to (3.1) and inequality (3.9) is equivalent to (3.7).

In particular, when \(p = q = 2\), we have

\[\int_0^\infty \left[ \int_0^\infty \left| \ln x \right| f(x) dx \right] dy < 144 \int_0^\infty f^2(x) dx,\]

and

\[\int_0^\infty \left[ \int_0^\infty \left| \frac{\ln x}{y} \right| f(x) dx \right] dy < 64 \int_0^\infty f^2(x) dx.\]

**Proof.** We set

\[g(y) = \left[ \int_0^\infty \frac{1 + \left| \ln \frac{x}{y} \right|}{\max \{x, y\}} f(x) dx \right]^{p-1}.\]

Then, by (3.1), we have

\[0 < \int_0^\infty g^q(y) dy = \int_0^\infty \left[ \int_0^\infty \frac{1 + \left| \ln \frac{x}{y} \right|}{\max \{x, y\}} f(x) dx \right] dy \]

\[= \int_0^\infty \int_0^\infty \frac{1 + \left| \ln \frac{x}{y} \right|}{\max \{x, y\}} f(x) g(y) dxdy \]

\[< \left[ p^2 + pq + q^2 \right] \left\{ \int_0^\infty f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(y) dy \right\}^{\frac{1}{q}}.\]

Hence, we get
\begin{align*}
0 < \left\{ \int_0^\infty g^q(y) dy \right\}^{1 \frac{1}{q}} & = \left\{ \int_0^\infty \left[ \int_0^\infty \frac{1 + \ln \frac{x}{y}}{\max\{x, y\}} f(x) dx \right] dy \right\}^{\frac{1}{p}} \\
& < \left[ p^2 + pq + q^2 \right] \left\{ \int_0^\infty f^p(x) dx \right\}^{\frac{1}{p}}.
\end{align*}

Therefore, inequality (3.8) holds.

On the other hand assume that (3.8) is valid. By Holder’s inequality, we obtain

\begin{align*}
\int_0^\infty \int_0^\infty \frac{1 + \ln \frac{x}{y}}{\max\{x, y\}} f(x) g(y) dx dy & = \int_0^\infty \left[ \int_0^\infty \frac{1 + \ln \frac{x}{y}}{\max\{x, y\}} f(x) dx \right] [g(y) dy] \\
& \leq \left\{ \int_0^\infty \left[ \int_0^\infty \frac{1 + \ln \frac{x}{y}}{\max\{x, y\}} f(x) dx \right]^{p} \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(y) dy \right\}^{\frac{1}{q}} \\
& \leq \left[ p^2 + pq + q^2 \right] \left\{ \int_0^\infty f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(y) dy \right\}^{\frac{1}{q}}.
\end{align*}

Then by (3.8) we obtain (3.1). Therefore, (3.8) and (3.1) are equivalent. If the constant factor \([p^2 + pq + q^2]^p\) in (3.8) is not the best possible, using (3.10) we may get a contradiction that the constant factor in (3.1) is not the best possible. The proof of (3.9) is the same. Thus the theorem is proved.

4. Discrete Analogous

**Theorem 4.1.** If \(p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_n \geq 0, b_n \geq 0\) such that \(0 < \sum_{n=1}^\infty a_n^p < \infty, 0 < \sum_{n=1}^\infty b_n^q < \infty\), we have

\begin{equation}
\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{1 + \ln \frac{m}{n}}{\max\{m, n\}} a_m b_n < \left[ p^2 + pq + q^2 \right] \left\{ \sum_{n=1}^\infty a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty b_n^q \right\}^{\frac{1}{q}},
\end{equation}

where the constant factor \([p^2 + pq + q^2]^p\) is the best possible. In particular, when \(p = q = 2\), we have
\[\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1 + |\ln \frac{m}{n}|}{\max \{m, n\}} a_m b_n < 12 \left(\sum_{n=1}^{\infty} a_n^2\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^2\right)^{\frac{1}{q}}.\]

**Proof.** Define the weight coefficient \(\tilde{\omega}(r, n)\) as

\[\tilde{\omega}(r, n) = \sum_{m=1}^{\infty} \frac{1 + |\ln \frac{n}{m}|}{\max \{m, n\}} \left(\frac{n}{m}\right)^{\frac{1}{r}}, \quad r = p, q; \quad n \in N.\]

Since the function \(f(u) = \frac{1 + |\ln \frac{n}{u}|}{\max \{u, n\}} \left(\frac{u}{n}\right)^{\frac{1}{r}}\) is decreasing, then we obtain

\[\tilde{\omega}(r, n) < \int_{0}^{\infty} \frac{1 + |\ln \frac{n}{u}|}{\max \{u, n\}} \left(\frac{n}{u}\right)^{\frac{1}{r}} \, du = p^2 + pq + q^2.\]

By Holder’s inequality and (4.2), following the method of proof in Theorem 3.1, we get:

\[\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1 + |\ln \frac{n}{m}|}{\max \{m, n\}} a_m b_n < \left\{\sum_{n=1}^{\infty} \tilde{\omega}(q, n) a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} \tilde{\omega}(p, n) b_n^q\right\}^{\frac{1}{q}}.\]

Using (4.3) we get (4.1). For \(0 < \varepsilon < p - 1\), setting \(\tilde{a}_m = n^{-\frac{1+\varepsilon}{p}}, \tilde{b}_n = n^{-\frac{1+\varepsilon}{q}}\), by (3.6) we get:

\[\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1 + |\ln \frac{n}{m}|}{\max \{m, n\}} \tilde{a}_m \tilde{b}_n = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1 + |\ln \frac{n}{m}|}{\max \{m, n\}} n^{-\frac{1+\varepsilon}{p}} n^{-\frac{1+\varepsilon}{q}} > \int_{1}^{\infty} \int_{1}^{\infty} \frac{1 + |\ln \frac{x}{y}|}{\max \{x, y\}} f(x) \tilde{g}(y) \, dx \, dy = \frac{1}{\varepsilon} \left(p^2 + pq + q^2 + o(1)\right).

On the other hand,

\[\left\{\sum_{n=1}^{\infty} \tilde{a}_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} \tilde{b}_n^q\right\}^{\frac{1}{q}} = \sum_{n=1}^{\infty} \frac{1}{n^{\varepsilon+1}} < 1 + \frac{1}{\varepsilon}.\]

By using the inequalities (4.4), (4.5) and the same technique in the proof of Theorem 3.1, we can show that the constant factor in (4.1) is the best possible.

**Corollary 4.2.** Under the same conditions in Theorem 4.1 we still have
\begin{equation}
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{|\ln \frac{m}{n}|}{\max \{m, n\}} a_m b_n \leq \left[ p^2 + q^2 \right] \left\{ \sum_{n=1}^{\infty} a_n^p \right\} \frac{1}{p} \left\{ \sum_{n=1}^{\infty} b_n^q \right\} \frac{1}{q},
\end{equation}
where the constant factor \( p^2 + q^2 \) is the best possible. In particular, when \( p = q = 2 \), we have

\begin{equation}
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{|\ln \frac{m}{n}|}{\max \{m, n\}} a_m b_n \leq 8 \left\{ \sum_{n=1}^{\infty} a_n^2 \right\} \frac{1}{2} \left\{ \sum_{n=1}^{\infty} b_n^2 \right\} \frac{1}{2}.
\end{equation}

\textbf{Proof.} Note that

\begin{equation}
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1 + |\ln \frac{m}{n}|}{\max \{m, n\}} a_m b_n = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max \{m, n\}} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{|\ln \frac{m}{n}|}{\max \{m, n\}} a_m b_n,
\end{equation}
using this equality, inequality (1.4) and applying the proof of Corollary 3.2, we can easily prove (4.7).

\textbf{Theorem 4.3.} Under the same conditions in Theorem 4.1 we have

\begin{equation}
\sum_{n=1}^{\infty} \left[ \sum_{m=1}^{\infty} \frac{1 + |\ln \frac{m}{n}|}{\max \{m, n\}} a_m \right]^p < [p^2 + pq + q^2]^p \sum_{n=1}^{\infty} a_n^p,
\end{equation}

\begin{equation}
\sum_{n=1}^{\infty} \left[ \sum_{m=1}^{\infty} \frac{|\ln \frac{m}{n}|}{\max \{m, n\}} a_m \right]^p \leq [p^2 + q^2]^p \sum_{n=1}^{\infty} a_n^p,
\end{equation}
where the constant factors \( p^2 + pq + q^2 \) and \( p^2 + q^2 \) are the best possible. Inequality (4.8) is equivalent to (4.1) and inequality (4.9) is equivalent to (4.6).

\textbf{Proof.}

Setting

\[ b_n = \left[ \sum_{m=1}^{\infty} \frac{1 + |\ln \frac{m}{n}|}{\max \{m, n\}} a_m \right]^{p-1}, \]
we get
$$\sum_{n=1}^{\infty} b_n^p = \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \frac{1 + |\ln \frac{m}{n}|}{\max \{m, n\}} a_m \right\}^p$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1 + |\ln \frac{m}{n}|}{\max \{m, n\}} a_m b_n.$$  

By (4.1) and using the same method of Theorem (3.3), we obtain (4.8). We may show that the constant factor in (4.8) is the best possible and inequality (4.1) is equivalent to (4.8). The proof of (4.9) is the same.

Remark 4.4. Observe that inequality (1.6) when combined with inequality (1.5), gives

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{|\ln \frac{m}{n}|}{\max \{m, n\}} a_m a_n \leq 8 \sum_{n=1}^{\infty} a_n^2,$$

in agreement with (4.7).

References


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