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# Behavior Threshold Conditions in SIS STD Models

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**Abstract**

In multi-group epidemiological models with nonrandom mixing between people in the different groups, often artificial constraints have to be imposed in order to satisfy the balance conditions. Based on the model in this article, we construct a simple biased mixing model where the balance conditions are automatically satisfied as a natural consequence of the equations. We propose and analyze a heterogeneous, multigroup, susceptible-infective-susceptible (SIS) sexually transmitted disease (STD) model where the desirability and acceptability in partnership formations are functions of the infected individuals.

**Mathematics Subject Classification:** 92BXX

**Keywords:** Balance constraint; Reproductive number; Sensitivity; Partners

## 1 Introduction

Sexually transmitted disease (STD), such as AIDS, have spread into nearly all countries of the world. To prevent further spread of these epidemics, it is important to understand how these infectious disease are transmitted.The transmission dynamics are complex. Many biological and sociological factors are involved. One of major determinants in the spread of STD's is the way that individuals select their sexual partners. In a mathematical model for the spread of AIDS, it is important to understand and correctly account for the formation of their partnerships. In modeling partnerships, the partnership formation must satisfy the balance constraints [2, 3, 4, 1, 7, 11, 15].

Sexual behavior changes are documented in virtually every survey of homosexual or bisexual men and injection drug users over the past decade [1, 6, 16, 17]. These behavior changes occur as sexually active individuals become more cautious in their sexual activities to avoid infection by an STD such as AIDS. Understanding the effect of these behavior changes can help guide education programs on the prevention of STD transmission.

Some of the analysis of these behavior studies [6, 13, 19], has implied that the reported behavior changes combined with observed reduction in incidence of rectal gonorrhea, HIV infection, and AIDS have been large enough to reduce the rate of transmission of HIV and possibly reduce the rate of HIV transmission below the epidemic threshold [9].

A goal of this research is to better understand how models with dynamic partnership formation differ from the more traditional models where the number or desirability of partnership formation is constant. By making the partnership formation infection dependent, we can analyze how sensitive the transmission dynamics of the epidemic are to changes in sexual behaviors.

## 2 Model Formation

Divide the susceptible and infected population into $K$ groups, $S_i$ and $I_i$, $i = 1, ..., K$, and consider the simple transmission model system,

$$\begin{cases} dS_i/dt = \mu(S_i^0 - S_i) - \lambda_i S_i + \gamma_i I_i, \\ dI_i/dt = -(\mu + \gamma_i) I_i + \lambda_i S_i, \end{cases} \quad i = 1, ..., K, \tag{1}$$

where $\mu$ is the natural death rate, $\gamma_i$ is the rate of recovery for infected individuals in group $i$, $\lambda_i$ is the rate of infection, and $\mu S_i^0$ is the rate of recruitment into group $i$.

The formation of partnerships plays an essential role in determining the function $\lambda_i$, which is one of the most important factors in modeling STD's. Here a partnership is an activity between two people where the infection can be transmitted. We assume people in each group behave the same when selecting a partner, but have biases between groups, but there is heterogeneous mixing among the groups.

Let $q_{ij}$ be the preference of individuals in group $i$ to have a partner from group $j$; that is, $q_{ij}$ the fraction of people in group $j$ with whom an individual in group $i$ desires forming a partnership. Thus $q_{ij}$ describes the desirability of individuals in group $i$ to have a partner from group $j$. It is also the acceptability of people in group $j$ to people in group $i$ [10].

Under the condition that enough potential partners are available, the probability $p_{ij}$ that a partnership forms between individuals from group $i$ and group $j$, or the mutually acceptable rate for partnership formation [14], is,

$$p_{ij} \equiv q_{ij} q_{ji}.$$

Define $c_i$ to be the number of social contacts per unit time for a person in group $i$. The probability that a contact is with a person from group $j$ is $c_j N_j / \sum_k c_k N_k$, where $N_k = S_k + I_k$. This also characterize the availability of sexual contacts with partners in group $j$. Hence, the probability of a partnership forming between individuals from group $i$ and group $j$ is $p_{ij} \left( c_j N_j / \sum_k c_k N_k \right)$ [9].

We define $\beta_{ij}$ to be the probability of disease transmission per contact between an infected partner in group $j$ and a susceptible individual in group $i$ [9, 10]. Under these assumptions, the infection rate of people in group $i$ is

$$\lambda_i = c_i \sum_{j=1}^{K} p_{ij} \beta_{ij} (c_j I_j / \sum_k c_k N_k), \tag{2}$$

where we assume that $I_i / N_j$ is the probability that a random contact from group $j$ is with an infected individual.

## 2.1 Balance constraints

We denote the number of partners of people in group $i$ from group $j$ by $T_{ij}$. Note that $T_{ij} = T_{ji}$. In many biased mixing models where an attempt is made to directly control the number of partnered by constructing preferred, selective, or structured mixing functions [7, 8, 11, 12, 18]. Therefore, it follows from

$$T_{ij} = p_{ij} (c_j N_j / \sum_k c_k N_k) c_i N_i = p_{ij} (c_i N_i / \sum_k c_k N_k) c_j N_j = T_{ji} \tag{3}$$

that the balance constraint is always satisfied. Using the advantages of the selective mixing model, we further assume that the desirability and acceptability depend on the fraction of infected individuals in the populations. This assumption characterizes possible behavior changes of sexually active individuals. More specifically, we assume that the desirability of people in group $i$ having a partner in group $j$ or the acceptability of people in group $j$ to people in group $i$, $q_{ij}$, is a decreasing function of the fraction of infected individuals in group $j$. Then the mutually acceptable rates for partnership formation can be expressed as,

$$p_{ij} = p_{ji} = q_{ij}(I_j / N_j) q_{ji}(I_i / N_i). \tag{4}$$

Therefore, the infection rates are

$$\begin{aligned}\lambda_i &= c_i \sum_{j=1}^{K} \beta_{ij} q_{ij} (I_j/N_j) q_{ji} (I_i/N_i)(c_j I_j / \sum_k c_k N_k) \\ &= c_i I_i / N_i \sum_{j=1}^{K} \beta_{ij} q_{ij} q_{ji} (I_j/N_j)(c_j I_j / \sum_k c_k N_k). \qquad (5)\end{aligned}$$

## 2.2 The number of partners

The number of sexual partners per individual in many multi-group models is assumed to be $c_i$ [10]. However,If the mixing is biased, the number of partners will vary in time depending on the combination of desirability, acceptability, and availability. From Section 2, the number of partners per person in group $i$ is

$$n_i = c_i \left( \sum_{j=1}^{K} p_{ij} (c_j N_j / \sum_k c_k N_k) \right). \qquad (6)$$

## 2.3 Example

Consider a two group model governed by

$$\begin{cases} dS_i/dt = \mu(S_i^0 - S_i) - \lambda_i S_i + \gamma_i I_i, \\ dI_i/dt = -(\mu + \gamma_i) I_i + \lambda_i S_i, \end{cases} \quad i = 1, 2, \qquad (7)$$

with

$$\begin{aligned}\lambda_i &= c_i \sum_{j=1}^{2} \beta_{ij} q_{ij} q_{ji} (c_j I_j^2 I_i / N_i N_j N^0) \\ &= (c_i \beta_{i1} q_{i1} q_{1i} ((c_1 I_1^2 I_i / N_i N_1 N^0)) + c_i \beta_{i2} q_{i2} q_{2i} ((c_2 I_2^2 I_i / N_i N_2 N^0))). \qquad (8)\end{aligned}$$

Then

$$n_i = c_i (p_{i1} (c_1 N_1 / (c_1 N_1 + c_2 N_2)) + p_{i2} (c_2 N_2 / (c_1 N_1 + c_2 N_2))), \qquad (9)$$

and

$$\begin{aligned} n_1 - n_2 &= (1/(c_1 N_1 + c_2 N_2))(c_1^2 p_{11} N_1 + c_1 c_2 p_{12} N_2 - c_2 c_1 p_{21} N_1 - c_2^2 p_{22} N_2) \\ &= (1/(c_1 N_1 + c_2 N_2))((c_1 p_{11} - c_2 p) c_1 N_1 + (c_1 p - c_2 p_{22}) c_2 N_2), \qquad (10)\end{aligned}$$

where $p \equiv p_{12} = p_{21}$. If $c_1 < c_2$ and $p_{11} < p < p_{22}$ or $c_1 > c_2$ and $p_{11} > p > p_{22}$, then $n_1$ is always less than or greater than $n_2$ respectively. Otherwise, they may alternate at different times. We use the following model parameters, $S_1^0 = 450, S_1(0) = 450, I_1(0) = 50, S_2^0 = 200, S_2(0) = 200, I_2(0) = 350, c_1 =$

$10, c_2 = 5, \mu = 0.015, q_{11} = 0.6, q_{12} = 1, q_{21} = 0.5, q_{22} = 0.2, \gamma_1 = 0.1, \gamma_2 = 0.05$. Depending on the probability of transmission, the disease may spread in the population or die out eventually. For example, when $\beta_{11} = 0.5, \beta_{12} = 0.4, \beta_{21} = 0.4, \beta_{22} = 0.2$, the disease dies out (Fig.1), but when $\beta_{11} = 0.5, \beta_{12} = 0.5, \beta_{21} = 0.4, \beta_{22} = 0.2$, the disease persists (Fig. 2).

# 3 Threshold Conditions

The contact of threshold conditions is one of the most important concepts in mathematical epidemiology. It specifies when the disease spreads if a small number of infected people are introduced into the susceptible population. The threshold conditions are usually characterized by the reproductive number which can be obtained by the study of stability of the infection-free equilibrium. In model (1), there is an infection-free equilibrium $(S_i = S_i^0, I_i = 0), i = 1, ..., K$. The stability of this equilibrium is completely determined by the equations of $I_i$ about the equilibrium $I_i = 0$, and can be investigated by either constructing a Liapunov function or examining the eigenvalues of the Jacobian matrix evaluated at the equilibrium [10].

## 3.1 Reproductive number

The jacobian matrix of (8) at the zero solution has the form of

$$J^0 = \begin{pmatrix} -\mu - \gamma_1 + \beta_{11} p_{11} S_1^0 c_1^2 / N^0 & \beta_{12} p_{12} S_1^0 c_1 c_2 / N^0 \\ \beta_{21} p_{21} S_2^0 c_1 c_2 / N^0 & -\mu - \gamma_2 + \beta_{22} p_{22} S_2^0 c_2^2 / N^0 \end{pmatrix}, \tag{11}$$

with $N^0 = \sum_{j=1}^{2} c_j S_j^0$ . We simplify the notation by defining $\delta_i = \mu + \gamma_i$ and $a_{ij} = \beta_{ij} S_i^0 c_i c_j / N^0$. Then

$$J^0 = \begin{pmatrix} -\delta_1 + a_{11} p_{11} & a_{12} p_{12} \\ a_{21} p_{21} & -\delta_2 + a_{22} p_{22} \end{pmatrix}. \tag{12}$$

The larger eigenvalue of $J^0$,

$$\lambda = (-(1/\delta) + (a_{11}p_{11} + a_{22}p_{22}) + \sqrt{((\delta_1 - a_{11}p_{11}) - (\delta_2 - a_{22}p_{22}))^2 + 4(a_{12}p_{12})(a_{21}p_{21})})/2, \tag{13}$$

with $\delta = 1/(\delta_1 + \delta_2)$ is real. If $\lambda < 0$, the zero solution of (8) is stable, and if $\lambda > 0$, it is unstable. Now ,we can take the reproductive number, $R_0$, as below,

$$R_0 = \delta(a_{11}p_{11} + a_{22}p_{22}) + \sqrt{((\delta_1 - a_{11}p_{11}) - (\delta_2 - a_{22}p_{22}))^2 + 4(a_{12}p_{12})(a_{21}p_{21})}, \tag{14}$$

with $\delta = 1/(\delta_1 + \delta_2)$. Hence, if $R_0 < 1$, the epidemic dies out,and if $R_0 > 1$, the epidemic spreads in the population.

### 3.1.1 Sensitivity studies

Consider, the two group model where the behavior of people in group 1, $(q_{12}, q_{11})$, and the average acceptability of place in group 2, $a = q_{22} + q_{21}$ are fixed. We now use $q_{21} \equiv p, 0 \leq p \leq a$, as a parameter to study the effects of the relative acceptability of people in group 1 on the reproductive number. A large $p$ implies that people in group 2 prefer their partners more from group 1 and are less interested in forming partners within their own group. In terms of $p$,

$$R_0(p) = \delta(a_{11}(a-p)^2+a_{22}q_{22}^2)+\sqrt{((\delta_1 - a_{11}(a-p)^2) - (\delta_2 - a_{22}q_{22}^2))^2 + (4a_{12}a_{21}q_{21}^2)p^2)}, \tag{15}$$

with $\delta = 1/(\delta_1 + \delta_2)$.

### 3.1.2 Example

For the two group model (8) ,when we use equation (16) with the parameters, $S_1^0 = 100, S_2^0 = 200, \beta_{11} = 0.12, \beta_{12} = 0.08, \beta_{21} = 0.05, \beta_{22} = 0.1, \mu = 0.015, \gamma_1 = 0.1, \gamma_2 = 0.05, q_{21} = 0.7$,
we have Fig.3, and the dynamics of the susceptibles and infecteds for different $p$'s are shown in Fig.4.

Now,we consider, the two group model where the behavior of people in group 2, $(q_{21}, q_{22})$, and the average acceptability of place in group 1, $b = q_{11}+q_{12}$ are fixed. We now use $q_{12} \equiv q, 0 \leq q \leq b$, as a parameter to study the effects of the relative acceptability of people in group 2 on the reproductive number. A large $q$ implies that people in group 1 prefer their partners more from group 2 and are less interested in forming partners within their own group. In terms of $q$,

$$R_0(q) = \delta(a_{11}q_{11}^2+a_{22}(b-q)^2)+\sqrt{((\delta_1 - a_{11}q_{11}^2) - (\delta_2 - a_{22}(b-q)^2))^2 + (4a_{12}a_{21}q_{12}^2)q^2)}, \tag{16}$$

with $\delta = 1/(\delta_1 + \delta_2)$.

### 3.1.3 Example

For the two group model (8) ,when we use equation (17) with the parameters, $S_1^0 = 100, S_2^0 = 200, \beta_{11} = 0.12, \beta_{12} = 0.08, \beta_{21} = 0.05, \beta_{22} = 0.1, \mu = 0.015, \gamma_1 = 0.1, \gamma_2 = 0.05, q_{21} = 0.7$,
we have Fig.5, and the dynamics of the susceptibles and infecteds for different $q$'s are shown in Fig.6.

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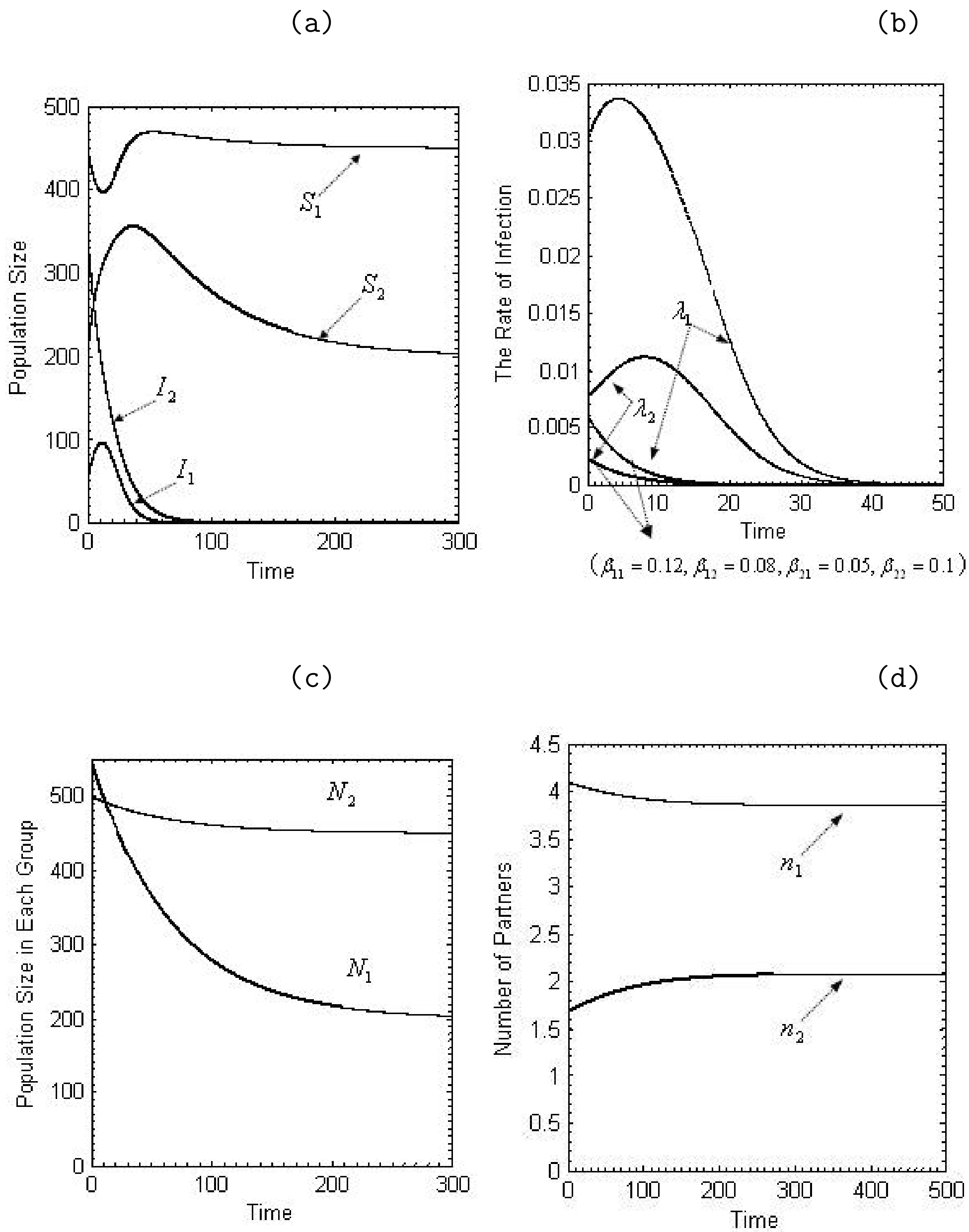


Figure 1: $S_1^0 = 450, S_1(0) = 450, I_1(0) = 50, S_2^0 = 200, S_2(0) = 200, I_2(0) = 350, c_1 = 10, c_2 = 5, \mu = 0.015, q_{11} = 0.6, q_{12} = 1, q_{21} = 0.5, q_{22} = 0.2, \gamma_1 = 0.1, \gamma_2 = 0.05, \beta_{11} = 0.5, \beta_{12} = 0.4, \beta_{21} = 0.4, \beta_{22} = 0.2.$

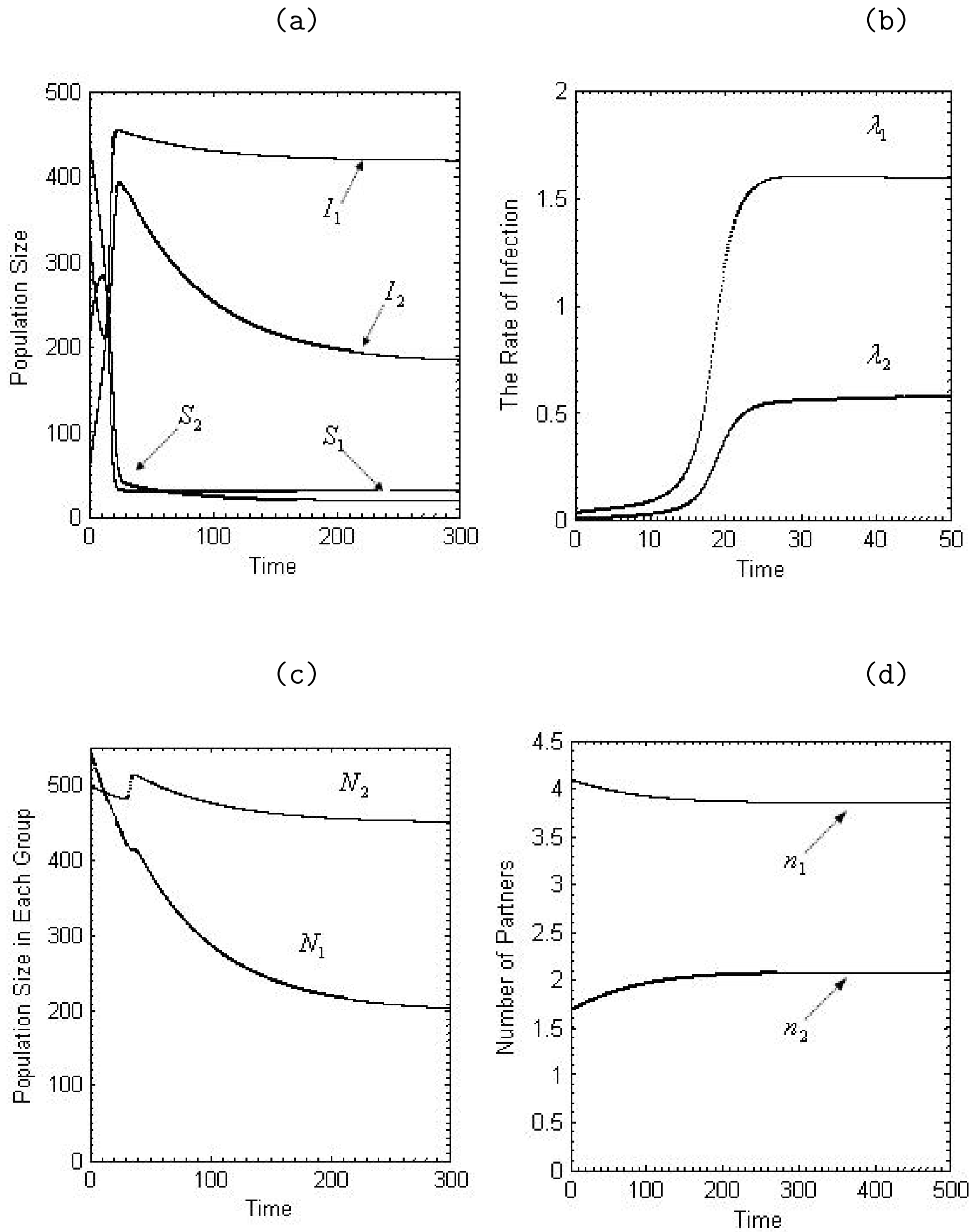


Figure 2: $S_1^0 = 450, S_1(0) = 450, I_1(0) = 50, S_2^0 = 200, S_2(0) = 200, I_2(0) = 350, c_1 = 10, c_2 = 5, \mu = 0.015, q_{11} = 0.6, q_{12} = 1, q_{21} = 0.5, q_{22} = 0.2, \gamma_1 = 0.1, \gamma_2 = 0.05, \beta_{11} = 0.5, \beta_{12} = 0.5, \beta_{21} = 0.4, \beta_{22} = 0.2.$

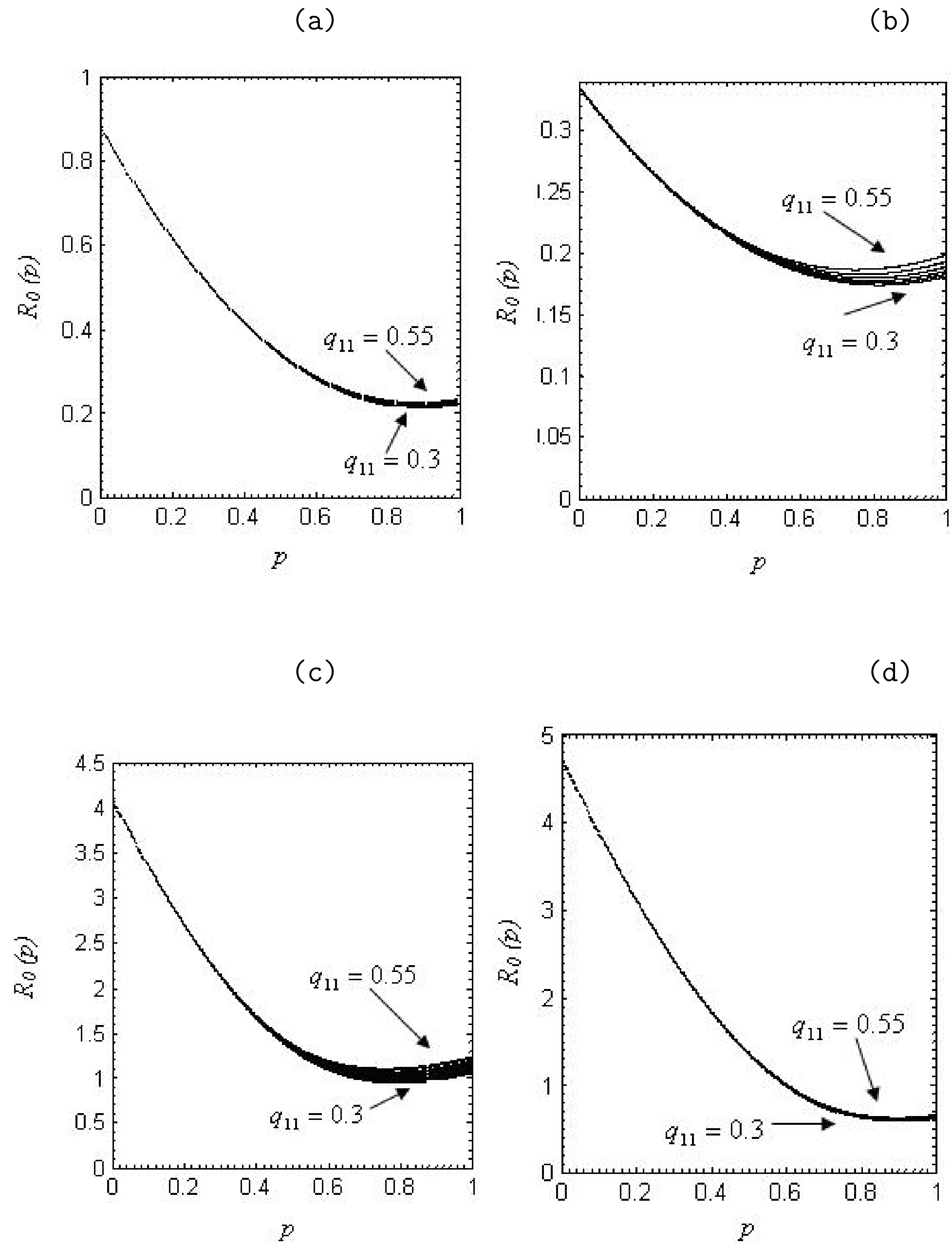


Figure 3: $S_1^0 = 100, S_2^0 = 200, \beta_{11} = 0.12, \beta_{12} = 0.08, \beta_{21} = 0.05, \beta_{22} = 0.1, \mu = 0.015, \gamma_1 = 0.1, \gamma_2 = 0.05, q_{21} = 0.7, (a) : c_1 = 10, c_2 = 10, (b) : c_1 = 10, c_2 = 4, (c) : c_1 = 8, c_2 = 10, (d) : c_1 = 4, c_2 = 10.$

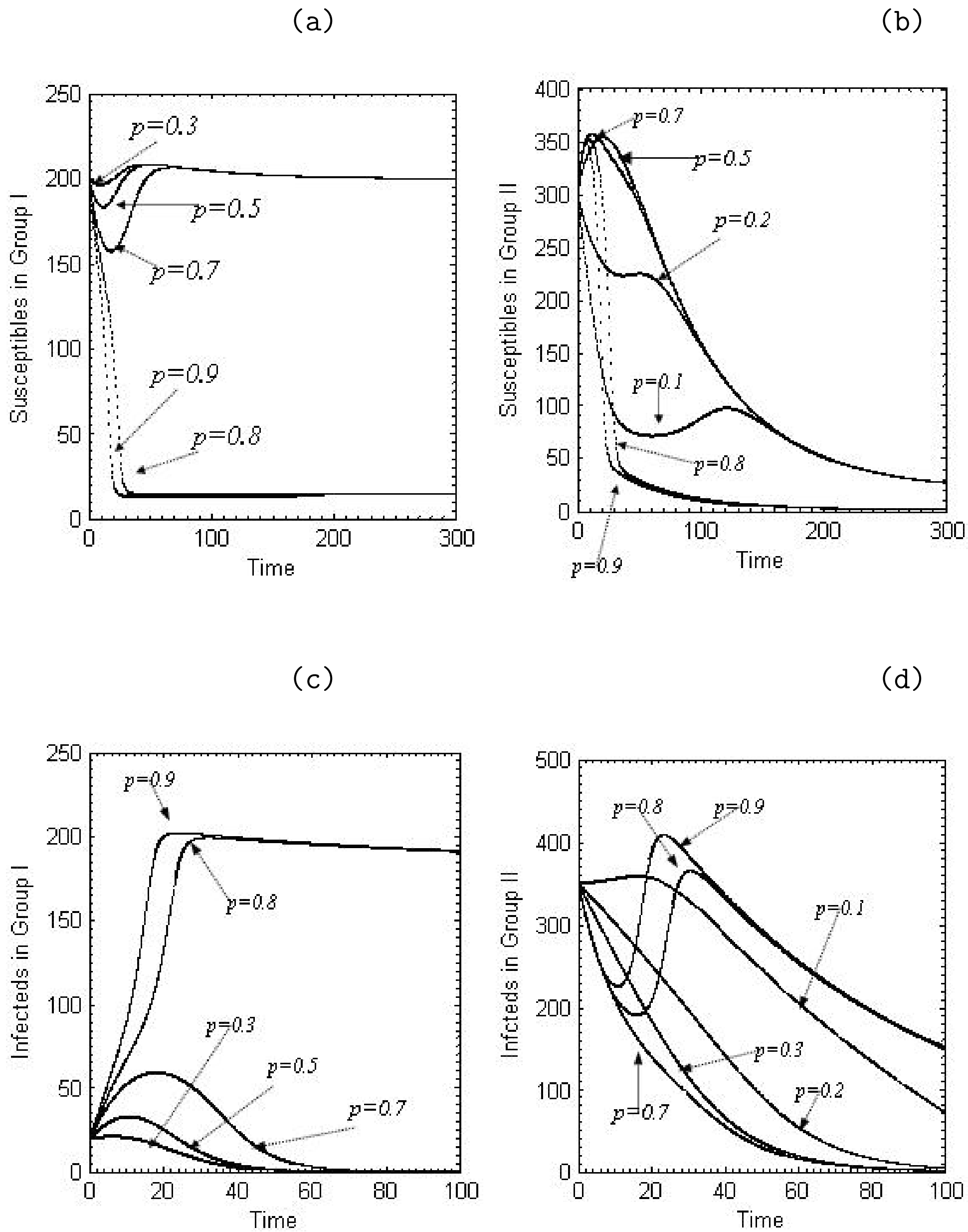


Figure 4: $S_1^0 = 100, S_2^0 = 200, \beta_{11} = 0.12, \beta_{12} = 0.08, \beta_{21} = 0.05, \beta_{22} = 0.1, \mu = 0.015, \gamma_1 = 0.1, \gamma_2 = 0.05, q_{12} = 0.7, q_{11} = 0.6, c_1 = 10, c_2 = 5.$

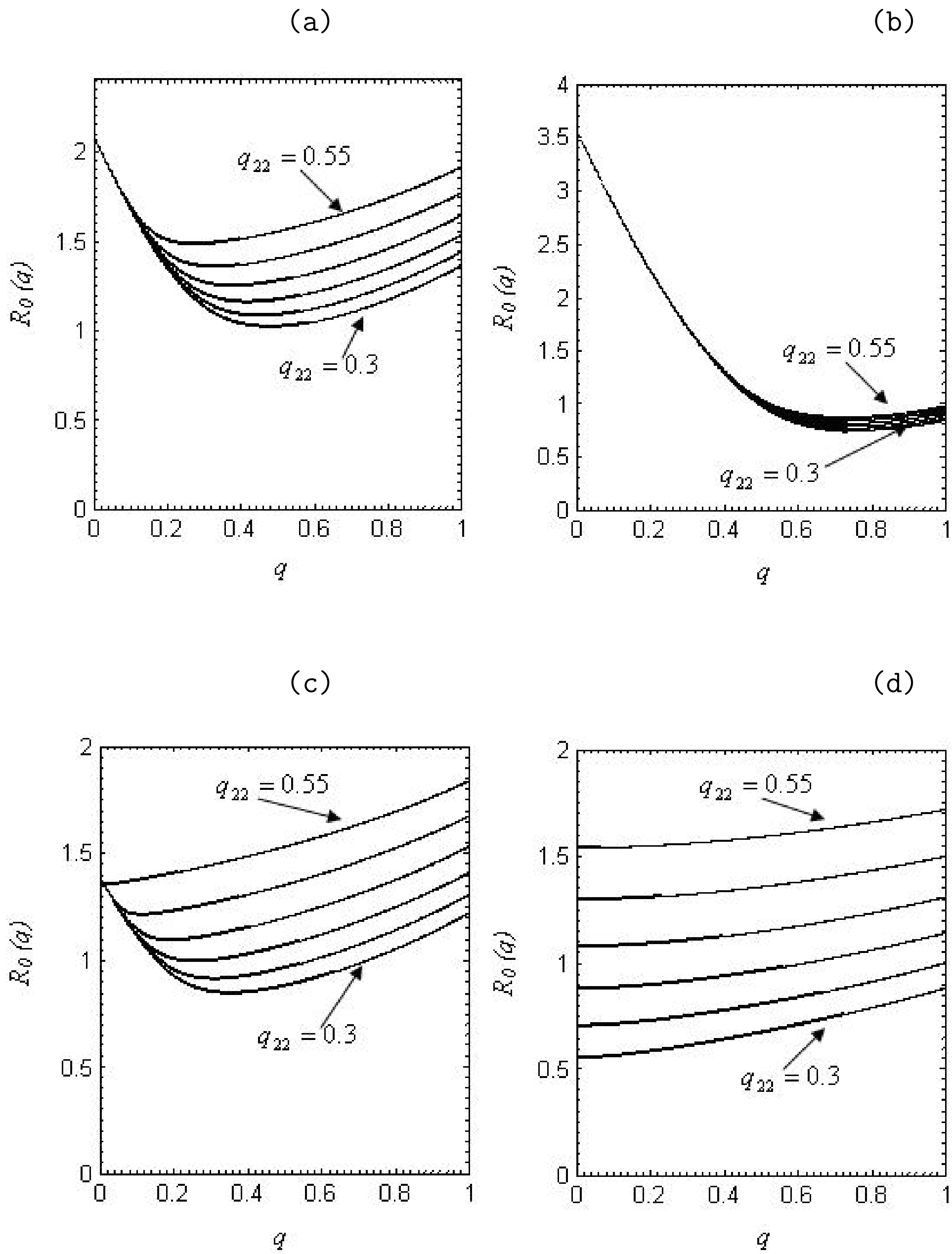


Figure 5: $S_1^0=100, S_2^0=200, \beta_{11}=0.12, \beta_{12}=0.08, \beta_{21}=0.05, \beta_{22}=0.1, \mu=0.015, \gamma_1=0.1, \gamma_2=0.05, q_{21}=0.7, (a): c_1=10, c_2=10, (b): c_1=10, c_2=4, (c): c_1=8, c_2=10, (d): c_1=4, c_2=10.$

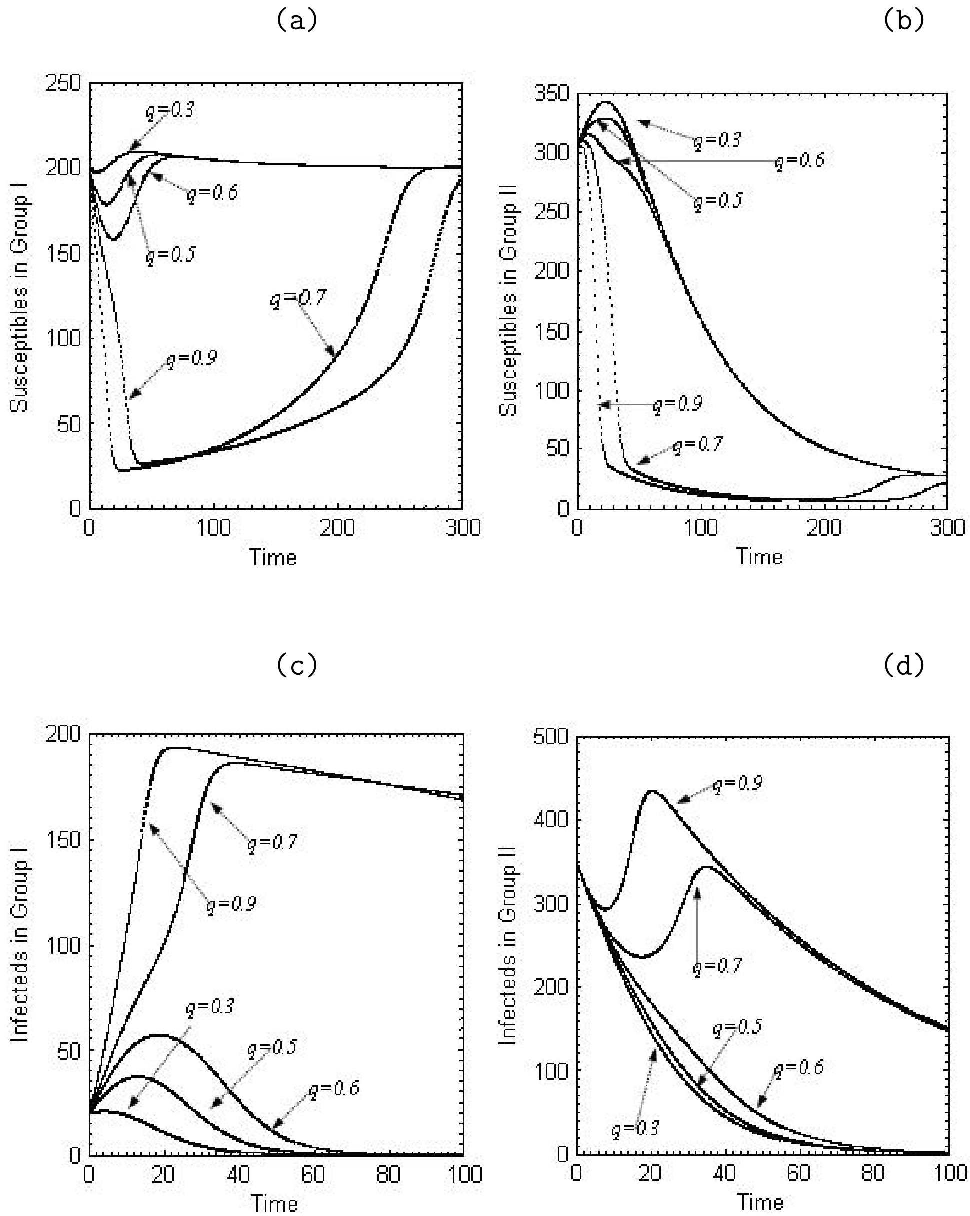


Figure 6: $S_1^0 = 100, S_2^0 = 200, \beta_{11} = 0.12, \beta_{12} = 0.08, \beta_{21} = 0.05, \beta_{22} = 0.1, \mu = 0.015, \gamma_1 = 0.1, \gamma_2 = 0.05, q_{21} = 0.6, q_{22} = 0.4, c_1 = 10, c_2 = 5.$