On Amenability of Certain Semigroup Algebras

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Abstract

In this paper, we study weak amenability and amenability of certain semigroup algebras of foundation topological semigroups.

Mathematics Subject Classification: Primary 43A20; Secondary 46H20

Keywords: Amenability; foundation semigroup; derivation; semigroup algebras; weak amenability

1 Introduction

Throughout, $S$ denotes a locally compact, Hausdroff topological semigroup, and $M(S)$ denotes the space of all bounded complex regular measures on $S$. This space with the convolution product $*$, and norm $||\mu|| = |\mu|(S)$ is a Banach algebra. The space of all measures $\mu \in M(S)$ for which the mappings $x \mapsto \delta_x*|\mu|$ and $x \mapsto |\mu|*\delta_x$ from $S$ into $M(S)$ are weakly continuous is denoted by $M_a(S)$ (or $\tilde{L}(S)$) as in [1], where $\delta_x$ denotes the Dirac measure at $x$. Note that the measure algebra $M_a(S)$ defines a two-sided closed $L$-ideal of $M(S)$ (see [1]). Denote by $L^\infty(S, M_a(S))$ the set of all complex-valued

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This paper isstracted from a research supported by Islamic Azad university, Mobareke branch.
bounded functions $g$ on $S$ that are $M_a(S)$-measurable; we identify functions in $L^\infty(S, M_a(S))$ that agree $\mu$-almost everywhere for all $\mu \in M_a(S)$.

Note that in the case where $S$ is discrete (resp. a locally compact group), $L^\infty(S, M_a(S))$ is equal to $\ell^\infty(S)$ (resp. $L^\infty(S)$). Observe that $L^\infty(S, M_a(S))$ with the complex conjugation as involution, the pointwise operations and the norm $\|\cdot\|_\infty$ is a commutative $C^*$-algebra with identity $1$, where $1$ denotes the constant function one on $S$ and $\|\cdot\|_\infty$ is defined by

$$\| g \|_\infty = \sup \{ \| g \|_{\infty, |\mu|} : \mu \in M_a(S) \} \quad (g \in L^\infty(S, M_a(S))),$$

where $\| \cdot \|_{\infty, |\mu|}$ denotes the essential supremum norm with respect to $|\mu|$. The equation

$$\tau(g)(\mu) := \mu(g) = \int_S g \, d\mu$$

defines a linear map $\tau$ of $L^\infty(S, M_a(S))$ into the continuous dual space $M_a(S)^*$ of $M_a(S)$. Recall from [3] that a semigroups $S$ is called a foundation semigroup; if $\cup \{ \text{supp}(\mu) : \mu \in M_a(S) \}$ is dense in $S$. Note that in the case where $S$ is a foundation semigroup with identity, for every $\mu \in M_a(S)$ both mappings $x \mapsto \delta_x | \mu |$ and $x \mapsto | \mu | \ast \delta_x$ from $S$ into $M_a(S)$ are norm continuous and $\tau$ is onto (see [10]). The second dual $M_a(S)^{**}$ of $M_a(S)$ is a Banach algebra with the first Arens product defined by the equations

$$\langle F \circ H, f \rangle = \langle F, Hf \rangle, \quad \langle Hf, \mu \rangle = \langle H, \mu \circ f \rangle$$

for all $F, H \in M_a(S)^{**}$, $f \in M_a(S)^*$, and $\mu \in M_a(S)$.

Let $A$ be Banach algebra, and let $X$ be a Banach $A$-bimodule. The space of continuous derivations from $A$ into $X$ is denoted by $Z^1(A, X)$, and the space of continuous inner derivations from $A$ into $X$ is denoted by $B^1(A, X)$. The first cohomology group of $A$ with coefficients in $X$ is defined to be the linear space $H^1(A, X) = Z^1(A, X)/B^1(A, X)$. Thus $H^1(A, X) = \{0\}$ if and only if each continuous derivation from $A$ into $X$ is inner. Recall that if $A$ be a Banach algebra, then its dual $A^*$ can be a Banach $A$–bimodule, with module actions defined by $(f.a)(b) = f(ab)$ and $(a.f)(b) = f(ba)$ for all $a, b \in A$ and $f \in A^*$. A Banach algebra $A$ is called weak amenable if $H^1(A, A^*) = \{0\}$. Also $A$ is called amenable if $H^1(A, X^*) = \{0\}$ for all Banach $A$-bimodule $X$.

In this paper, we study weak amenability and amenability of $M(S)$ and its closed ideals; in particular $M_a(S)$ of certain foundation semigroups $S$.

## 2 The results

It is well-known that for any locally compact group $G$ the group algebra $L^1(G)$ is weakly amenable. The following example shows that this result is not true
even for a subsemigroup of a discrete group. We will need the following well-known result.

**Proposition 2.1** If a Banach algebra $A$ is such that $A^2 \neq A$ then $A$ is not weakly amenable. In particular if $S^2 \neq S$ for a discrete semigroup $S$ then $\ell^1(S)$ is not weakly amenable.

**Example 2.2** Let $S = (\mathbb{N}, +)$. We have $\ell^1(\mathbb{N})$ is not weakly amenable. In fact; since $\mathbb{N}^2 = \mathbb{N} + \mathbb{N} \neq \mathbb{N}$, then by Proposition 2.1, $\ell^1(\mathbb{N})$ is not weakly amenable.

For a locally compact semigroup $S$, let $M_0(S) = \{ \mu \in M(S) : \mu(S) = 0 \}$ be the augmentation of $M(S)$. It is immediate that $M_0(S)$ is indeed an ideal of $M(S)$.

**Proposition 2.3** Let $S$ be a commutative semigroup with identity. if $M(S)$ is weakly amenable, then so is $M_0(S)$.

**Proof.** Since $S$ is a semigroup with identity, from Proposition 2.1 of [9], it follows that $M_0(S)^2 = M_0(S)$. Now the proof is complete by Theorem 2.8.69 of [2].

**Proposition 2.4** Let $S$ be a commutative foundation semigroup with identity. Let $M_a(S)$ is weakly amenable, then any one co-dimensional ideal $I$ of $M_a(S)$ is weakly amenable.

**Proof.** In this case from Proposition 2.3 of [9], it follows that $I^2 = I$. Now the proof is complete by by Theorem 2.8.69 of [2].

Recall that $LUC(S)$ be the space of all function $g \in C_b(S)$, that the mapping $x \mapsto \tau g$ from $S$ into $C_b(S)$ is $\|\cdot\|_\infty$-continuous, where $C_b(S)$ denotes the space of all bounded continuous complex-valued functions on $S$, and $\tau g(y) = g(xy)$ for all $x, y \in S$. The following result gives a generalization of Theorem 2.1 from [4]. We note that $M(S)$ is not weak amenable when $S$ is a locally compact group, in general.

**Theorem 2.5** Let $S$ be a foundation semigroup with identity such that $C^{-1}D$ and $CD^{-1}$ is a compact subset of $S$ for every tow compact subset $C$ and $D$ of $S$. Then weakly amenable of $M_a(S)^{**}$ implies weak amenability of $M(S)$.

**Proof.** First we note that by Lemma 1 of [6] $C_0(S)^\perp$ is a closed ideal of $LUC(S)^*$ with $LUC(S)^* = M(S) \oplus C_0(S)^\perp$, where $C_0(S)$ is the subset of $C_b(S)$ consisting of functions vanishing at infinity. Also, since $M_a(S)$ has a bounded approximate identity with norm one (see [5]), there exists a right identity $E$
of $M_a(S)^{**}$ with $\|E\| = 1$. Now, from ([8], page 1198) and Lemma 2.1 of [5], it follows that $EM_a(S)^{**}$ is isometrically isomorphic with $LUC(S)^*$ and so

$$M_a(S)^{**} = LUC(S)^* + (I - E)M_a(S)^{**}$$

where $I$ is the identity map on $M_a(S)^{**}$ and $(I - E)M_a(S)^{**}$ is a closed ideal having trivial product. Now suppose that $M(S)$ is not weakly amenable, and let $D : M(S) \to M(S)^*$ be a non-inner derivation. Define $\Delta : LUC(S)^* \to LUC(S)^{**}$ by

$$\Delta(\mu + h) = T_{D(\mu)},$$

where $T_{D(\mu)} \in LUC(S)^{**}$ is given by

$$\langle T_{D(\mu)}, \nu + h \rangle = \langle D(\mu), \nu \rangle \quad (\nu \in M(S), h \in C_0(S)^\perp).$$

Since $C_0(S)^\perp$ is an ideal in $LUC(S)^*$, $\Delta$ is a derivation. If there is a $\Phi \in LUC(S)^{**}$ with $\Delta = ad_\Phi$, then $\Psi = \Phi|_{M(S)}$ is an element of $M(S)^*$ with $D = ad_\Psi$. Now the derivation $\Lambda : M_a(S)^{**} \to M_a(S)^{***}$ given by

$$\Lambda(n) = \Delta(En) \quad (n \in M_a(S)^{**})$$

is not inner. This is a contradiction. □

By Theorem 1.3 of [4], it follows that $M_a(S)^{**}$ is amenable for any finite semigroup $S$. In the following we show that the converse is true for certain locally compact semigroups. Before, we note that

$$Z_t(M_a(S)^{**}) = \{ H \in M_a(S)^{**} : F \mapsto H \circ F \text{ is weak}^* - \text{weak}^* \text{ continuous} \}.$$

**Theorem 2.6** Let $S$ be a cancellative foundation $*-\text{semigroup}$ with identity such that $C^{-1}D$ is a compact subset of $S$ for every tow compact subset $C$ and $D$ of $S$. Then $M_a(S)^{**}$ is amenable if and only if $S$ is finite.

**Proof.** It is a well-known result that $M_a(S)^{**}$ has a bounded approximate identity when $M_a(S)^{**}$ be amenable. Thus by Lemma 2.1 from [4] it follows that $M_a(S)^{**}$ has an identity $E$. Clearly that the map $F \mapsto E \circ F$ on $M_a(S)^{**}$ is weak$^*$-weak$^*$ continuous, so $E \in Z_t(M_a(S)^{**})$. Now, by Corollary 3.2 of [7] we have $E \in M_a(S)$, so that $S$ must therefore be discrete. The result is now an immediate consequence of Theorem 1.3 of [4]. □

**Corollary 2.7** Let $S$ be a open subsemigroup with identity of a locally compact group $G$. Then $M(S)^{**}$ is amenable if and only if $S$ is finite.
Proof. We know that \( M_a(S) = M(S) \) for finite semigroup \( S \), so the “if” part is clear. For the converse, it is not hard to see that \( M_a(S) \) is complemented in \( M(S) \), and so \( M_a(S)^{**} \) is complemented in \( M(S)^{**} \). Moreover, \( M_a(S)^{**} \) is an ideal in \( M(S)^{**} \), and so itself amenable if \( M(S)^{**} \) is amenable. Now the proof is complete by Theorem 2.6. \( \square \)

Acknowledgments. The authors would like to thank the referee of the paper for invaluable comments and wish to thank the Azad university of Mobarake for its support.

References


Received: September 16, 2008