Some Remarks on Multiplication and Comultiplication Modules

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Abstract

This paper deals with some results concerning multiplication and comultiplication modules over a commutative ring.

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1 Introduction

Throughout this paper $R$ will denote a commutative ring with identity. Also for an $R$-module $M$, the notation $\text{grade}(I, M)$ will denote the grade $I$ relative to $M$, where $R$ is a commutative Noetherian ring and $I$ is an ideal of $R$. We will follow the terminology concerning $\text{grade}(I, M)$ and Cohen-Macaulay modules from [4].

An $R$-module $M$ is called a multiplication module if for every submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = IM$.

An ideal $I$ of $R$ is said to be second (see [7]) if for each $r \in R$, we have $rI = 0$ or $rI = I$.

A submodule $N$ of an $R$-module $M$ is said to be completely irreducible (see [6]) if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of submodules of $M$, then $N = N_i$ for some $i \in I$.

A submodule $N$ of an $R$-module $M$ is said to be large (see [1]) if for every non-zero submodule $L$ of $M$, $N \cap L \neq 0$.

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An $R$-module $M$ is said to be cocyclic (see [6]) if $\text{Soc}(M)$ is large and a simple submodule of $M$.

The sets $\text{Ass}_R(M)$ and $\text{Supp}_R(M)$ are defined as

$$\text{Ass}_R(M) = \{ P \in \text{Spec}(R) : P = (0:R \ x), \text{for some non-zero element } x \text{ of } M \};$$

$$\text{Supp}_R(M) = \{ P \in \text{Spec}(R) : P \supseteq (0:R \ x), \text{for some non-zero element } x \text{ of } M \}.$$

In [2], the dual notion of multiplication modules was introduced and the first properties of this class of modules have been considered. We recall that $M$ is a comultiplication module (see [2]) if for every submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = (0 :_M I)$. Also it is shown that (see [2, 3.7]) $M$ is a comultiplication module if and only if for each submodule $N$ of $M$, $N = (0 :_M \text{Ann}_R(N))$.

2 Main results

Remark 2.1 (see [3]). Let $M$ be a comultiplication $R$-module. Then

(a) If $P$ is a maximal ideal of $R$ and $(0 :_M P) \neq 0$, then $(0 :_M P)$ is simple.

(b) If $B$ is an ideal of $R$ such that $(0 :_M B) = 0$, then for every element $m \in M$, there exists an element $b$ of $B$ such that $m = bm$.

Theorem 2.2. Let $M$ be a faithful multiplication $R$-module. Then we have the following.

(a) If $R$ is a Noetherian ring and $I$ is an ideal of $R$, then $\text{grade}(I, M) = \text{grade}(I, R)$.

(b) If $R$ is a Noetherian ring, then $\text{Ass}_R(M) = \text{Ass}_R(R)$.

(c) If $M$ is a finitely generated semisimple module, then $R$ is a semisimple ring.

Proof. (a) First note that since $R$ is Noetherian ring, $M$ is a finitely generated module by [5]. It is enough to show that every sequence in $I$ is an $R$-sequence if and only if it is an $M$-sequence. To see this, let $I$ be an ideal of $R$ and let $X := x_1, x_2, \ldots, x_n$ be an $R$-sequence in $I$. By [5, 3.1], $XR \neq R$ if and only if $XM \neq M$. Now assume that $x_i m \in (x_1, \ldots, x_{i-1})M$, where $1 \leq i \leq n$ and $m \in M$. Since $M$ is a multiplication $R$-module, there exists an ideal $J$ of $R$ such that $Rm = JM$. Thus $x_i JM \subseteq (x_1, \ldots, x_{i-1})M$. Hence $x_i J \subseteq (x_1, \ldots, x_{i-1})$ by
[5, 3.1]. Therefore, \( J \subseteq (x_1, \ldots, x_{i-1}) \). So \( Rm = JM \subseteq (x_1, \ldots, x_{i-1})M \). This implies that \( m \in (x_1, \ldots, x_{i-1})M \). It follows that \( X \) is an \( M \)-sequence. The reverse implication is proved similarly.

(b) Let \( P \in \text{Ass}_R(M) \). Then \( P = (0 :_R m) \) for some \( m \in M \). Since \( M \) is a multiplication \( R \)-module, there exists an ideal \( I \) of \( R \) such that \( Rm = IM \). Thus \( P = (0 :_R IM) \). Since \( M \) is faithful, \( P = (0 :_R I) \). Since \( R \) is Noetherian there exists \( a \in I \) such that \( P = (0 :_R a) \). Thus \( \text{Ass}_R(M) \subseteq \text{Ass}_R(R) \).

Corollary 2.3. Let \( R \) ba a Noetherian ring and let \( M \) be a faithful multiplication \( R \)-module. Then \( M \) is a Cohen-Macaulay \( R \)-module if and only if \( R \) is a Cohen-Macaulay ring.

Theorem 2.4. Let \( U \) be a comultiplication \( R \)-module. Then

(a) If \( N \) is a finitely cogenerated submodule of \( U \), then there exists a finitely generated ideal \( I \) of \( R \) such that \( N = (0 :_U I) \).

(b) \( \sum_{f \in M^*} \text{Im} f = (0 :_U \text{Ann}_R(M^*)) \), where \( M^* = \text{Hom}_R(M, U) \).

(c) \( \text{Max}(R) \cap A(U) \subseteq \text{Ass}_R(U) \), where

\[
A(U) = \{ P \in \text{Spec}(R) : (0 :_U P) \neq 0 \}.
\]

(d) \( \text{Supp}_R(U) \subseteq A(U) \).

Proof. (a) Let \( L \) be a completely irreducible submodule of \( U \). Then \( L = (0 :_U I) = \cap_{a \in I} (0 :_U a) \), where \( I = \text{Ann}_R(L) \). Thus \( L = (0 :_U a) \) for some \( a \in I \). Now since \( N \) is finitely cogenerated, \( N = \bigcap_{i=1}^n L_i \), where \( L_i \) is a completely irreducible submodule of \( U \) for each \( i \). Therefore, \( N = \bigcap_{i=1}^n (0 :_U a_i) \) for some \( a \in \text{Ann}_R(L_i) \). Thus \( N = (0 :_U I) \), where \( I = Ra_1 + Ra_2 + \ldots + Ra_n \).

(b) Let \( V = \sum_{f \in M^*} \text{Im} f \). Then \( V \) is a submodule of \( U \), and hence \( V = (0 :_U I) \) for some ideal \( I \) of \( R \). Let \( \theta \in M^* \). Then \( \theta(M) \subseteq V \). This implies that
Thus \( I \theta(M) = 0 \). Thus \((I \theta)M = 0\). It follows that \( I \theta = 0 \). Hence \( I \subseteq \text{Ann}_R(M^*) \) and \((0 :_U \text{Ann}_R(M^*)) \subseteq V\). On the other hand, for any \( \phi \in M^* \),

\[
\phi(\text{Ann}_R(M^*)M) = \text{Ann}_R(M^*)\phi(M) = (\text{Ann}_R(M^*)\phi)M = 0
\]

Thus \( \text{Ann}_R(M^*)\phi(M) = 0 \). It follows that \( V \subseteq (0 :_U \text{Ann}_R(M^*)) \) as desired.

(c) Suppose that \( P \in \text{Max}(R) \cap A(U) \). Then \((0 :_U P) \neq 0 \) and it is a minimal submodule of \( U \) by Remark 2.1 (a). Hence there exists \( 0 \neq m \in U \) such that \((0 :_U P) = Rm \) so that \( P \subseteq \text{Ann}_R(Rm) \). Since \( P \) is maximal and \( 0 \neq m, P = \text{Ann}_R(Rm) \) as desired.

(d) Suppose that \( P \in \text{Supp}(U) \). Then there exists \( 0 \neq m \in U \) such that \((0 :_R m) \subseteq P\). Assume that \((0 :_U P) = 0 \). Then by Remark 2.1 (b), there exists \( p \in P \) such that \((1 - p)m = 0\). Hence \((1 - p) \in (0 :_R m) \subseteq P\), a contradiction. Therefore, \((0 :_U P) \neq 0\) as desired.

**Theorem 2.5.**

(a) \( M \) be a non-zero multiplication \( R \)-module and let \( S \) be a second ideal of \( R \) such that \( SM = M \). Then \( M \) is a cocyclic \( R \)-module.

(b) Let \( R \) be a ring which is not a filed and let \( S_1 \) and \( S_2 \) be simple \( R \)-modules such that \( S_1 + S_2 \) is faithful. Then \( S_1 + S_2 \) is a comultiplication \( R \)-modules.

(c) Let \( R \) be a Noetherian ring and let \( M \) be a faithful divisible multiplication \( R \)-module. Then \( R \) is a semi-local ring.

Proof. (a) By [6], \( M \) has a proper completely irreducible submodule \( L \). Since \( M \) is a multiplication \( R \)-module, \( L = IM \) for some ideal \( I \) of \( R \). Thus \( L = IM = SIM \). Since \( S \) is second, \( SI = 0 \) or \( SI = S \). Hence \( L = 0 \) or \( L = M \). Since \( L \) is proper, \( L = 0 \). Therefore, \( M \) is a cocyclic \( R \)-module.

(b) Let \( M = S_1 + S_2 \). It is clear that \( S_1 \subseteq (0 :_M \text{Ann}_R(S_1)) \). Suppose that \( m \in (0 :_M \text{Ann}_R(S_1)) \). Then \( m = m_1 + m_2 \) where \( m_1 \in S_1 \) and \( m_2 \in S_2 \) and \( m\text{Ann}_R(S_1) = 0 \). If \( m \notin S_1 \), then \( m_2 \notin S_1 \). Since \( m_2\text{Ann}_R(S_1) = 0 \), we have \( m_2 \in S_2 \cap (0 :_M \text{Ann}_R(S_1)) \). This in turn implies that \( m_2 = 0 \) or \( S_2 \subseteq (0 :_M \text{Ann}_R(S_1)) \). Hence \( M = S_1 + S_2 \subseteq (0 :_M \text{Ann}_R(S_1)) \). Thus \( \text{Ann}_R(S_1) = 0 \). But this is a contradiction, because \( R \) is not a filed. Therefore, \( m \in S_1 \) as desired.

(c) Let \( m \) be a maximal ideal of \( R \). Since \( R \) is Noetherian, \( M \) is finitely generated by [5]. Thus by [5, 3.1], \( mM \neq M \). Now since \( M \) is divisible, \( m \in Zd(R) = \bigcup_{P \in \text{Ass}(R)} P \). Thus \( m \in \text{Ass}(R) \). This implies that \( R \) has a finite
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number of maximal ideals and the proof is completed.

References


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