Idealization and Primary Decomposition

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Abstract

Let $R$ be a commutative ring with identity and $M$ an $R$-module. In this paper, we study the primary decomposition of the ideals of the ring $R(+M)$.

Mathematics Subject Classification: 13B99; 13A15

Keywords: Idealization; Primary decomposition

1 Introduction and preliminaries

Throughout this paper, let $R$ denote a commutative ring with identity and all modules are assumed to be unitary. Let $M$ be an $R$-module; the idealization $R(+M)$ (also called the trivial extension) introduced by Nagata in [5], and several authors have extended this concept, see for example [1]. Recall that $R(+M)$ with addition

$$(r_1(+m_1) + (r_2(+m_2)) = (r_1 + r_2(+m_1) + m_2),$$

and multiplication

$$(r_1(+m_1)(r_2(+m_2)) = (r_1r_2(+r_1m_2 + r_2m_1),$$

is a commutative ring with identity, called the idealization of $M$. Note that $R$ naturally embeds into $R(+M)$ via $r \rightarrow r(+)0$, if $N$ is a submodule of $M$, then $0(+)N$ is an ideal of $R(+M)$, $0(+)M$ is a nilpotent ideal of $R(+M)$ of index 2, every ideal that contains $0(+)M$ has the form $I(+)M$ for some ideal $I$ of $R$, and every ideal that is contained in $0(+)N$ has the form $0(+)K$ for some submodule $K$ of $N$. Some basic results on idealization can be found in [3].
In this paper, we focus on primary decomposition of ideals of the ring $R(+)M$. Let $Q$ be an ideal of $R$. We say that $Q$ is a primary ideal of $R$ precisely if $Q \subseteq R$, that is $Q$ is a proper ideal of $R$ and whenever $a, b \in R$ with $ab \in Q$ but $a \notin Q$ then there exists $n \in N$ such that $b^n \in Q$ equivalently, if $a, b \in R$ and $ab \in Q$ imply $a \in Q$ or $b \in \sqrt{Q}$ where $\sqrt{Q}$ denotes radical of $Q$. Let $I$ be a proper ideal of $R$. A primary decomposition of $I$ is an expression for $I$ as intersection of finitely many primary ideals of $R$, such a primary decomposition $I = Q_1 \cap Q_2 \cap \ldots \cap Q_n$ with $\sqrt{Q_i} = p_i$ for $i = 1 \ldots n$ of $I$, (usually it is said that $Q_i$ is a $p_i$-primary ideal, for all $i = 1 \ldots n$, whenever we use this type of terminology) is said to be a minimal primary decomposition of $I$ precisely when $p_1 \ldots p_n$ are $n$ different prime ideals of $R$, and for all $j$ with $1 \leq j \leq n$, $\cap_{i=1,i\neq j}^n Q_i$ is not contained in $Q_j$. We say that $I$ is decomposable ideal of $R$ if it has a primary decomposition.

1.1 Main Results

Primary decomposition is a basic problem in the studying of commutative rings. Here, we study primary decomposition of ideals of the ring $R(+)M$.

Let $M$ be an $R$-module. It is well-known that any prime (maximal) ideal of the ring $R(+)M$ is of the form $p(+)M$ such that $p$ is a prime (maximal) ideal of $R$.

**Theorem 1.1** Let $I$ be an ideal of $R$. Then $I$ is a decomposable ideal of $R$ if and only if $I(+)M$ is a decomposable ideal of $R(+)M$.

Proof. Suppose that $I$ is a decomposable ideal of $R$, and $I = Q_1 \cap Q_2 \cap \ldots \cap Q_n$ is a primary decomposition of $I$ in which $Q_i$ is a $p_i$-primary for all $i$. Then, $I(+)M = (Q_1 \cap \ldots \cap Q_n)(+)M = (Q_1(+)M) \cap \ldots \cap (Q_n(+)M)$.

Let $(a(+)m_1)(b(+)m_2) \in Q_i(+)M$ and $a(+)m_1 \notin Q_i(+)M$. Thus $ab(+)am_2 + bm_1 \in Q_i(+)M$ and $a \notin Q_i$, so there exists $n \in N$ such that $b^n \in Q_i$, because $Q_i$ is a primary ideal of $R$. Now, $(b(+)m_2)^n = (b^n(+)nm_2b^{n-1}) \in (Q_i(+)M)$, and so $Q_i(+)M$ is a primary ideal of $R(+)M$. An straightforward proof shows that $\sqrt{Q_i(+)M} \subseteq p_i(+)M$. Also if $t(+)m \in p_i(+)M$, then $t \in p_i = \sqrt{Q_i}$, and it means that $t^{k(+)km} = (t(+)m)^k \in Q_i(+)M$, for some positive integer $k$. Thus $t(+)m \in \sqrt{Q_i(+)M}$, and hence $\sqrt{Q_i(+)M} = p_i(+)M$. So $I(+)M$ is a decomposable ideal of ring $R(+)M$. Conversely, let $I(+)M$ have a primary decomposition of the form $I(+)M = Q_1' \cap \ldots \cap Q_n'$ with $\sqrt{Q_i} = p_i(+)M$, in which $Q_i'$s are $p_i(+)M$-primary ideals of the $R(+)M$. The well-known isomorphism $R \cong R(+)M/0(+)M$ gives a one-to-one correspondence between ideals of $R$ and ideals of $R(+)M$ that contains $0(+)M$. Clearly, $0(+)M \subseteq I(+)M \subseteq Q_i'$. So we have $Q_i = Q_i(+)M$ in which every $Q_i$ is a $p_i$-primary ideal of $R$. Hence $I(+)M = (Q_1(+)M) \cap (Q_2(+)M) \cap \ldots \cap (Q_n(+)M)$.

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\[ (Q_n(+M) = (Q_1 \cap Q_2 \cap \ldots \cap Q_n)(+)M, \text{ and so } I = Q_1 \cap Q_2 \cap \ldots \cap Q_n, \text{ as desired.} \]

**Corollary 1.2** If \( Q_1, Q_2, \ldots, Q_n \). \( (n \geq 1) \) are \( p \)-primary ideals of \( R \), then 
\[ \cap_{i=1}^n(Q_i(+M) \text{ is a } p(+M) \text{-primary ideal of } R(+)M. \]

Proof. It is clear by Theorem 1 and [6, lemma 4.13].

Note that if a proper ideal \( I(+)M \) of \( R(+)M \) has a primary decomposition with \( t \) terms which is not minimal, then \( I(+)M \) has a minimal primary decomposition with fewer than \( t \) terms. Minimal primary decompositions have certain uniqueness properties.

The following corollary follows from Theorem 1, and [6, Theorem 4.17].

**Corollary 1.3** Let \( I \) be a decomposable ideal of \( R \), and let \( I = Q_1 \cap Q_2 \cap \ldots \cap Q_n \) with \( \sqrt{Q_i} = p_i \) for \( i = 1 \ldots n \), be a minimal primary decomposition of \( I \). Let \( P \in Spec(R) \). Then \( I(+)M \) has primary decomposition and the following statements are equivalent:

(i) \( P(+M) = P_i(+M) \), for some \( i \) with \( 1 \leq i \leq n \);

(ii) There exists \( a(+)m \in R(+)M \) such that \( (I(+)M:a(+)m) \) is \( P(+M) \)-primary;

(iii) There exists \( a(+)m \in R(+)M \) such that \( \sqrt{(I(+)M : a(+)m)} = P(+M) \).

**Example 1.4** Let \( R = K[X,Y] \) be a ring of polynomials over the field \( K \) in indeterminates \( X, Y \) and \( M = K[X,Y] \). Consider the ideal \( I(+)M = (XY, Y^2)(+)M \) of \( R(+)M \). Theorem 1 shows that \( Q(+)M = (X, Y^2)(+)M \) and \( P(+M) = (Y)(+)M \) are primary ideals of the ring \( R(+)M \). One can easily check that \( I(+)M = (Q(+)M \cap (P(+)M) \). So \( I(+)M \) is a decomposable ideal of \( R(+)M \).

An ideal \( H \) of the idealization ring \( R(+)M \) is said to be homogeneous if \( H = I(+)N \), for some ideal \( I \) of \( R \) and a submodule \( N \) of \( M \). In this case, \( I(+)N = (R(+)M)(I(+)N) = I(+)IM + N \) gives \( IM \subseteq N \).

**Theorem 1.5** Let \( M \) be a finitely generated divisible module over a Noetherian integral domain \( R \). Then, \( R(+)M \) is Noetherian.

proof. Following [2, Theorem 3.3], if \( R \) is an integral domain, then every ideal of \( R(+)M \) is homogeneous if and only if \( M \) is divisible. Now consider the chain \( I_1(+)N(1) \subseteq I_2(+)N(2) \subseteq \ldots \) of ideals of \( R(+)M \). Then we have two chains \( I_1 \subseteq I_2 \subseteq \ldots \) of ideals of \( R \), and \( N_1 \subseteq N_2 \subseteq \ldots \) of submodule of \( M \). Since \( R \) is a Noetherian ring, there exists a positive integer \( n \) such that \( I_n = I_{n+i} \), for every \( i \geq 1 \) and since \( M \) is a Noetherian module, there exists a positive integer \( m \) such that \( N_m = N_{m+i} \), for every \( i \geq 1 \). Set \( t = \text{max}\{m, n\} \). Hence \( I_t = I_{t+i} \) and \( N_t = N_{t+i} \). Therefore \( I_t(+N(t)) = I_{t+i}(+)N_{(t+i)} \), for every \( i \geq 1 \). Hence \( R(+)M \) is Noetherian.
Corollary 1.6 Let $R$ be a Noetherian integral domain. If $M$ is a finitely generated divisible $R$-module, then every proper ideal of $R(+)M$ has a primary decomposition.

Proof. The result follows from Theorem 3, and [6].

The converse of the preceding theorem is presented in the following theorem.

Theorem 1.7 Let $R(+)M$ be Noetherian. Then $R$ is a Noetherian ring and $M$ is a Noetherian $R$-module.

Proof. Suppose that $N_1 \subseteq N_2 \subseteq \ldots$ is a chain of submodules of $M$. Then this chain induces the chain $0(+)N_1 \subseteq 0(+)N_2 \subseteq \ldots$ of ideals of $R(+)M$, so there exists $t \in N$ such that $0(+)N(t) = 0(+)N_1$, for every $i \geq 1$. Thus $N_i = N(t+i)_i$, for every $i \geq 1$. So $M$ is a Noetherian $R$-module. Since $R \cong R(+)M/0(+)M$ thus $R$ is a Noetherian ring.

Example 1.8 An example of a ring $R(+)M$ in which every ideal is homogeneous is $Z(+)Q$, where $Q$ is the field of rational numbers.

Following [4], a submodule $Q$ of $M$ is said to be a primary submodule of $M$ if $M \neq Q$ and for each zero-divisor $a$ of $M/Q$, there exists a positive integer $n$ such that $a^n(M/Q) = 0$.

Clearly, if $Q$ is a primary submodule of $M$, then $P := \sqrt{Ann_R(M/Q)}$ is a prime ideal in $R$, and in this case, we say that $Q$ is a $P$-primary submodule of $M$. Let $M$ be a module over a commutative ring $R$, and $G$ a proper submodule of $M$. A primary decomposition of $G$ in $M$ is an expression for $G$ as an intersection of finitely many primary submodules of $M$. We say that $G$ is a decomposable submodule of $M$ precisely when it has a primary decomposition in $M$. If we define $r_i(r(+)m) = r_1r(+)r_1m$, then $R(+)M$ will have an algebra structure.

Theorem 1.9 Let $N$ be a decomposable submodule of $M$. Then $R(+)N$ is a decomposable submodule of $R$-module $R(+)M$.

Proof. Let $N = Q_1 \cap Q_2 \cap \ldots Q_n$ be a decomposition of $N$ in which every $Q_i$ is a $p_i$-primary ideal. It is easy to check that

$$R(+)N = R(+)Q_1 \cap \ldots \cap Q_n = (R(+)Q_1) \cap \ldots \cap (R(+)Q_n).$$

First we know that $R(+)M \neq R(+)Q_i$, because $M \neq Q_i$. Second if $a \in Zdv_R(R(+)M/R(+)Q_i)$, then there exists $r(+)m \in R(+)M - R(+)Q_i$, such that $a(r(+)m + R(+)Q_i) = 0$. Hence $ar(+)am \in R(+)Q_i$, and so $am \in Q_i$ and $m \notin Q_i$; therefore, $a^n(M/Q_i) = 0$, for some positive integer $n$. Let $t(+)m \in R(+)M$. Then $a^n(t(+)m) = a^n(t)am \in R(+)Q_i$, and hence $a^n(R(+)M/R(+)Q_i) = 0$. It is clear that $\sqrt{ann(R(+)M/R(+)Q_i)} = p_i$; therefore, $R(+)Q_i$ is a $p_i$-primary submodule of $R(+)M$. 


Example 1.10 Let \( R = K[X, Y] \) be a ring of polynomials over the field \( K \) with indeterminates \( X, Y \), and \( K[X, Y](+)K[X, Y] \) be idealization of \( R \). Then \( T = K[X, Y](+)K[X, Y] \) \((X^3, XY)\) is a decomposable submodule of \( K[X, Y](+)K[X, Y] \) and
\[
K[X, Y](+)K[X, Y] = (K[X, Y](+)K[X, Y]) \cap (K[X, Y](+)K[X, Y])
\]
is a minimal primary decomposition of \( T \).

References


Received: March, 2009