Intuitionistic Fuzzy Sets in Ordered Semigroups

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Abstract

We consider the intuitionistic fuzzification of the concept of several ideals in an ordered semigroup $S$, and investigate some properties of such ideals.

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1 Introduction

After the introduction of fuzzy sets by L. A. Zadeh [8], several researches were conducted on the generalizations of the notion of fuzzy set. The concept of intuitionistic fuzzy set was introduced by K. T. Atanassov [1], as a generalization of the notion of fuzzy set. In [4], N. Kuroki gave some properties of fuzzy ideals and fuzzy semiprime ideals in semigroups. In this paper, we consider the intuitionistic fuzzification of the concept of several ideals in an ordered semigroup $S$, and investigate some properties of such ideals.

2 Preliminaries

We include some elementary aspects of ordered semigroups that are necessary for this paper.

By an ordered semigroup we mean an ordered set $S$ at the same time a semigroup satisfying the following conditions:

$$(\forall a, b, x \in S)(a \leq b \Rightarrow xa \leq xb \text{ and } ax \leq bx)$$

Let $(S, \cdot, \leq)$ be an ordered semigroup. A non-empty subset $U$ of $S$ is called a subsemigroup of $S$ if $U^2 \subseteq U$. 
A non-empty subset $A$ of an ordered semigroup $S$ is called a left (resp. right) ideal of $S$ if it satisfies:
- $SA \subseteq A$ (resp. $AS \subseteq A$),
- $(\forall a \in A)(\forall b \in S)(b \leq a \Rightarrow b \in A)$.

Both a left and a right ideal of $S$ is said to be ideal of $S$. A non-empty subset $A$ of an ordered semigroup $S$ is called a bi-ideal of $S$ if it satisfies:
- $ASA \subseteq A$,
- $(\forall a \in A)(\forall b \in S)(b \leq a \Rightarrow b \in A)$.

A subsemigroup $A$ of an ordered semigroup $S$ is called an interior ideal of $S$ if it satisfies:
- $SAS \subseteq A$,
- $(\forall a \in A)(\forall b \in S)(b \leq a \Rightarrow b \in A)$.

An ordered semigroup $S$ is called left-zero (resp. right-zero) if $x \leq xy$ (resp. $y \leq xy$) for all $x, y \in S$. An ordered semigroup $S$ is said to be left (resp. right) simple if for every left (resp. right) ideal $A$ of $S$, we have $A = S$. An ordered semigroup $S$ is said to be regular if for every $a \in S$ there exists $x \in S$ such that $a \leq axa$. $L[x]$ denote the principal left ideal of a semigroup $S$ generated by $x$ in $S$, that is, $L[x] = (x \cup Sx)$. By a fuzzy set $\mu$ in a non-empty set $X$, we mean a function $\mu : X \to [0, 1]$ and the complement of $\mu$, denoted by $\tilde{\mu}$, is the fuzzy set in $X$ given by $\tilde{\mu}(x) = 1 - \mu(x)$ for all $x \in X$. For any fuzzy subset $\mu$ in $S$ and $t \in [0, 1]$, we define

$$U(\mu; t) = \{ x \in S \mid \mu(x) \geq t \},$$

which is called an upper $t$-level cut of $\mu$ and can be used to the characterization of $\mu$.

An intuitionistic fuzzy set (briefly, IFS) $A$ in a nonempty set $X$ is an object having the form

$$A = \{ (x, \mu_A(x), \gamma_A(x)) \mid x \in X \}$$

where the functions $\mu_A : X \to [0, 1]$ and $\gamma_A : X \to [0, 1]$ denote the degree of membership and the degree of nonmembership, respectively, and

$$0 \leq \mu_A(x) + \gamma_A(x) \leq 1$$

for all $x \in X$. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the IFS $A = \{ (x, \mu_A(x), \gamma_A(x)) \mid x \in X \}$.

Let $\chi_U$ denote the characteristic function of a nonempty subset $U$ of an ordered semigroup.

**Definition 2.1.** Let $S$ be an ordered semigroup. A fuzzy set $\mu$ is called a fuzzy subsemigroup of $S$ if

$$\mu(xy) \geq \min\{\mu(x), \mu(y)\}$$

for all $x, y \in S$. 
Definition 2.2. Let \((S, \cdot, \leq)\) be an ordered semigroup. A fuzzy subsemigroup \(\mu\) of \(S\) is called a \textit{fuzzy bi-ideal} of \(S\), if the following axioms are satisfied:

\begin{enumerate}
\item[(FB1)] If \(x \leq y\), then \(\mu(x) \geq \mu(y)\),
\item[(FB2)] \(\mu(xy) \geq \min\{\mu(x), \mu(y)\}\), for all \(a, x, y \in S\).
\end{enumerate}

Example 2.3. Let \(S\) be a semigroup with the multiplication table:

\[
\begin{array}{ccc}
  & a & b & c \\
 a & c & b & c \\
b & b & b & c \\
c & c & c & c \\
\end{array}
\]

Let \(\leq:=\{(a, a), (b, b), (c, c)\}\). Then it is easy to verify that \((S, \cdot, \leq)\) is an ordered semigroup. Define a fuzzy subset \(f\) as follows: \(\mu(a) = \mu(b) = 0, \mu(c) = 0.3\).

By routine calculation, we know that \(\mu\) is a fuzzy bi-ideal of an ordered semigroup \(S\).

Proposition 2.4. Let \((S, \cdot, \leq)\) be an ordered semigroup and \(\phi \neq U \subseteq S\). Then \(U\) is a bi-ideal of \(S\) if and only if the characteristic function \(\chi_U\) is a fuzzy bi-ideal of \(S\).

Proof. Let \(x, s, y \in S\). If \(x, y \in U\) and \(s \in S\), then \(\chi_U(x) = \chi_U(y) = 1\). Since \(\chi_U\) is a bi-ideal of \(S\), we have \(x sy \in USU \subseteq U\), i.e. \(x sy \in U\). Thus \(\chi_U(x sy) = 1 = \min\{\chi_U(x), \chi_U(y)\}\). If \(x \notin U\) or \(y \notin U\), then \(\chi_U(x sy) \geq 0 = \min\{\chi_U(x), \chi_U(y)\}\) for all \(s \in S\). Let \(x, y \in S\) be such that \(x \leq y\). If \(y \in U\), then \(x \in U\). Thus \(\chi_U(x) = \chi_U(y) = 1\). If \(y \notin U\), then \(\chi_U(x) \geq \chi_U(y) = 0\). Hence \(\chi_U\) is a fuzzy bi-ideal of \(S\). Conversely, Let \(x, y \in U\) and \(s \in S\). Since \(\chi_U\) is a fuzzy bi-ideal of \(S\), we have \(\chi_U(x sy) \geq \min\{\chi_U(x), \chi_U(y)\}\), and so \(\chi(x sy) \geq 1\), i.e. \(x sy \in U\). Hence we have \(USU \subseteq U\). Let \(x, y \in S\) be such that \(x \leq y\) and \(y \in U\). Then \(\chi_U(x) \geq \chi_U(y) = 1\). Thus \(\chi_U(x) = 1\), i.e. \(x \in U\). This completes the proof. \(\square\)

3 Main results

In what follows, we use \(S\) to denote an ordered semigroup unless otherwise specified.

Definition 3.1. For an IFSS \(A = (\mu_A, \gamma_A)\) in \(S\), consider the following axioms:

\begin{enumerate}
\item[(IS1)] \(\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}\),
\item[(IS2)] \(\gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\}\), \(\forall x, y \in S\).
\end{enumerate}

Then \(A = (\mu_A, \gamma_A)\) is called a \textit{first} (resp. \textit{second}) \textit{intuitionistic fuzzy subsemigroup} (briefly, \(IFSS_1\) (resp. \(IFSS_2\))) of \(S\) if it satisfies \((IS1)\) (resp. \((IS2)\)). Also, \(A = (\mu_A, \gamma_A)\) is said to be an \textit{intuitionistic fuzzy subsemigroup} (briefly, \(IFSS\)) of \(S\) if it is both a first and a second intuitionistic fuzzy subsemigroup.
Theorem 3.2. If $U$ is a subsemigroup of ordered semigroup $S$, then $\tilde{U} = (\chi_U, \tilde{\chi}_U)$ is an IFSS of $S$.

Proof. Let $x, y \in S$. From the hypothesis, $xy \in U$ if $x, y \in U$. In this case,

$$\chi_U(xy) = 1 \geq \min \{\chi_U(x), \chi_U(y)\}$$

and

$$\tilde{\chi}_U(xy) = 1 - \chi_U(xy) \leq 1 - \min \{\chi_U(x), \chi_U(y)\} = \max \{1 - \chi_U(x), 1 - \chi_U(y)\} = \max \{\tilde{\chi}_U(x), \tilde{\chi}_U(y)\}.$$ 

If $x \notin U$ or $y \notin U$, then $\chi_U(x) = 0$ or $\chi_U(y) = 0$. Thus

$$\chi_U(xy) \geq 0 = \min \{\chi_U(x), \chi_U(y)\}$$

and

$$\max \{\tilde{\chi}_U(x), \tilde{\chi}_U(y)\} = \max \{1 - \chi_U(x), 1 - \chi_U(y)\} = 1 - \min \{\chi_U(x), \chi_U(y)\}$$

$$= 1 \geq \tilde{\chi}_U(xy).$$

This completes the proof. \qed

Theorem 3.3. Let $U$ be a non-empty subset of $S$. If $\tilde{U} = (\chi_U, \tilde{\chi}_U)$ is an IFSS$_1$ or IFSS$_2$ of $S$, then $U$ is a subsemigroup of an ordered semigroup $S$.

Proof. Suppose that $\tilde{U} = (\chi_U, \tilde{\chi}_U)$ is an IFSS$_1$ of $S$ and $x \in U^2$. In this case, $x = uv$ for some $u, v \in U$. It follows from (IS$_1$) that

$$\chi_U(x) = \chi_U(uv) \geq \min \{\chi_U(u), \chi_U(v)\} = 1.$$ 

Hence $\chi_U(x) = 1$, i.e. $x \in U$. Thus U is a subsemigroup of $S$. Now, assume that $\tilde{U} = (\chi_U, \tilde{\chi}_U)$ is an IFSS$_2$ of S and $x' \in U^2$. Then $x' = u'v'$ for some $u', v' \in U$. Using (IS$_2$), we get that

$$\tilde{\chi}_U(x') = \tilde{\chi}_U(u'v') \leq \max \{\tilde{\chi}_U(u'), \tilde{\chi}_U(v')\} = \max \{1 - \chi_U(u'), 1 - \chi_U(v')\} = 0$$

and so $\tilde{\chi}_U(x') = 1 - \chi_U(x') = 0$. Therefore $\chi_U(x') = 1$, i.e. $x' \in U$. This completes the proof. \qed

Definition 3.4. For an IFS $A = (\mu_A, \gamma_A)$ in $S$, consider the following axioms:

(II$_1$) $x \leq y$ implies $\mu_A(x) \geq \mu_A(y)$ and $\mu_A(xy) \geq \mu_A(y)$,

(II$_2$) $x \leq y$ implies $\gamma_A(x) \leq \gamma_A(y)$ and $\gamma_A(xy) \leq \gamma_A(y)$, $\forall x, y \in S$.

Then $A = (\mu_A, \gamma_A)$ is called a first (resp. second) intuitionistic fuzzy left ideal (briefly, IFIL$_1$ (resp. IFIL$_2$)) of $S$ if it satisfies (II$_1$) (resp. (II$_2$)). Also, $A = (\mu_A, \gamma_A)$ is said to be an intuitionistic fuzzy left ideal (briefly, IFIL) of $S$ if it is both a first and a second intuitionistic fuzzy left ideal.
Definition 3.5. For an IFS $A = (\mu_A, \gamma_A)$ in $S$, consider the following axioms:

(IR$_1$) $x \leq y$ implies $\mu_A(x) \geq \mu_A(y)$ and $\mu_A(xy) \geq \mu_A(x)$,

(IR$_2$) $x \leq y$ implies $\gamma_A(x) \leq \gamma_A(y)$ and $\gamma_A(xy) \leq \gamma_A(x)$, for all $x, y \in S$.

Then $A = (\mu_A, \gamma_A)$ is called a first (resp. second) intuitionistic fuzzy right ideal (briefly, IFRI$_1$ (resp. IFRI$_2$)) of $S$ if it satisfies (IR$_1$) (resp. (IR$_2$)). Also, $A = (\mu_A, \gamma_A)$ is said to be an intuitionistic fuzzy left ideal (briefly, IFLI) of $S$ if it is both a first and a second intuitionistic fuzzy right ideal.

Definition 3.6. Let $A = (\mu_A, \gamma_A)$ be an IFS in $S$. Then $A = (\mu_A, \gamma_A)$ is called an intuitionistic fuzzy ideal of $S$ if it is both an intuitionistic fuzzy left and an intuitionistic fuzzy right ideal.

Proposition 3.7. Let $U$ be a left-zero subsemigroup of $S$. If $A = (\mu_A, \gamma_A)$ is an IFLI of $S$, then the restriction of $A$ to $U$ is constant, that is, $A(x) = A(y)$ for all $x, y \in S$.

Proof. Let $x, y \in U$. Since $U$ is left-zero, $x \leq xy$ and $y \leq yx$. In this case, from the hypothesis, we have

$$\mu_A(x) \geq \mu_A(xy) \geq \mu_A(y), \quad \mu_A(y) \geq \mu_A(yx) \geq \mu_A(x)$$

and

$$\gamma_A(x) \leq \gamma_A(xy) \leq \gamma_A(y), \quad \gamma_A(y) \leq \gamma_A(yx) \leq \gamma_A(x).$$

Thus we obtain $\mu_A(x) = \mu_A(y)$ and $\gamma_A(x) = \gamma_A(y)$ for all $x, y \in U$. Hence $A(x) = A(y)$. \hfill \Box

Lemma 3.8. If $U$ is a left ideal of $S$, then $\tilde{U} = (\chi_U, \tilde{\chi}_U)$ is an IFLI of $S$.

Proof. Let $x, y \in S$ be such that $x \leq y$. Since $U$ is a left ideal of $S$, $x \in U$ and $xy \in U$ if $y \in U$. It follows that $x \leq y$ implies $\chi_U(x) = 1 = \chi_U(y)$ and $\tilde{\chi}_U(x) = 1 - \chi_U(x) = 0 = 1 - \chi_U(y) = \tilde{\chi}_U(y)$. Also, we have $\chi_U(xy) = 1 = \chi_U(y)$ and $\tilde{\chi}_U(xy) = 1 - \chi_U(xy) = 0 = 1 - \chi_U(y) = \tilde{\chi}_U(y)$.

If $y \notin U$, then $\chi_U(y) = 0$. In this case, $x \leq y$ implies $\chi_U(x) \geq 0 = \chi_U(y)$ and $\tilde{\chi}_U(x) \leq \tilde{\chi}_U(y) = 1 - \chi_U(y) = 1$. Also, we obtain $\chi_U(xy) \geq 0 = \chi_U(y)$ and $\tilde{\chi}_U(xy) = 1 - \chi_U(xy) = 1 \geq \tilde{\chi}_U(xy)$. Consequently, $\tilde{U} = (\chi_U, \tilde{\chi}_U)$ is an IFLI of $S$. \hfill \Box

An element $e$ in an ordered semigroup $S$ is called an idempotent if $ee \geq e$. Let $E_S$ denote the set of all idempotents in an ordered semigroup $S$.

Theorem 3.9. Let $A = (\mu_A, \gamma_A)$ be an IFLI of $S$. If $E_S$ is a left-zero subsemigroup of $S$, then $A(e) = A(e')$ for all $e, e' \in E_S$. 


Remark 3.1. It is clear that fuzzy interior ideal of $S$. Then

$$L_S = \mu_A(e) \geq \mu_A(ee') \geq \mu_A(e') \geq \mu_A(e)$$

and

$$\gamma_A(e) \leq \gamma_A(ee') \leq \gamma_A(e'), \quad \gamma_A(e') \leq \gamma_A(ee') \leq \gamma_A(e).$$

Hence we have $\mu_A(e) = \mu_A(e')$ and $\nu(e) = \nu(e')$ for all $e, e' \in E_S$. This completes the proof. \qed

Theorem 3.10. Let $S$ be regular. If, for every non-empty subset $U$ of $S$, $\tilde{U} = (\chi_U, \tilde{\chi}_U)$ is an $IFII_1$ (or $IFII_2$) of $S$ and $\tilde{U}(e) = \tilde{U}(e')$ for all $e, e' \in E_S$, then $E_S$ is a left-zero subsemigroup of $S$.

Proof. Since $S$ is regular, $E_S$ is non-empty. Let $e \leq ee, e' \leq ee' \in E_S$. Because $S$ is regular, $L[e] = (e \cup Se)$. Since $L[e]$ is a left ideal of $S$, we obtain $\tilde{L}[e] = (\chi_{L[e]}, \tilde{\chi}_{L[e]})$ is an $IFII_1$ (or $IFII_2$) of $S$ by Lemma 3.8. In this case, from the hypothesis, we get that

$$\chi_{L[e]}(e') = \chi_{L[e]}(e) = 1 \quad (or \quad \tilde{\chi}_{L[e]}(e') = \tilde{\chi}_{L[e]}(e) = 0).$$

Hence $e' \in L[e] = (e \cup Se)$. Thus $e' = e$ or $e' \in Se$. If $e \in Se$, we have $e' = xe \leq xee = (xe)e = e'e$ for some $x \in S$. If $e' = e$, we obtain $e' \leq e'e' = e'e$. This completes the proof. \qed

Definition 3.11. Let $(S, \cdot, \leq)$ be an ordered semigroup. A fuzzy subsemigroup $\mu$ of $S$ is called a fuzzy interior ideal of $S$, if the following axioms are satisfied:

1. $\mu(xay) \geq \mu(a)$,
2. If $x \leq y$, then $\mu(x) \geq \mu(y)$ for all $a, x, y \in S$.

Definition 3.12. For an IFS $A = (\mu_A, \gamma_A)$ in $S$, consider the following axioms:

1. $(II_1)$ $x \leq y$ implies $\mu_A(x) \geq \mu_A(y)$ and $\mu_A(xsy) \geq \mu_A(s)$,
2. $(II_2)$ $x \leq y$ implies $\gamma_A(x) \leq \gamma_A(y)$, and $\gamma_A(xsy) \leq \gamma_A(s), \forall x, y \in S$.

Then $A = (\mu_A, \gamma_A)$ is called a first (resp. second) intuitionistic fuzzy interior ideal (briefly, $IFII_1$ (resp. $IFII_2$)) of $S$ if it is an IFS$_1$ (resp. IFS$_2$) satisfying $(II_1)$ (resp. $(II_2)$). Also, $A = (\mu_A, \gamma_A)$ is said to be an intuitionistic fuzzy interior ideal (briefly, $IFII$) of $S$ if it is both a first and a second intuitionistic fuzzy interior ideal of $S$.

Remark 3.1. It is clear that $IFI$ of $S$ is an $IFII$ of $S$.

Theorem 3.13. If $S$ is regular, then every $IFII$ of $S$ is an $IFI$ of $S$. 
Proof. Let $A = (\mu_A, \gamma_A)$ be an IFII of $S$ and $x, y \in S$. In this case, because $S$ is regular, there exist $s, s' \in S$ such that $x s x \geq x$ and $y \leq y s' y$. Thus

$$\mu_A(xy) = \mu_A(x y s' y) = \mu_A(x y (s' y)) \geq \mu_A(y)$$

and

$$\gamma_A(xy) = \gamma_A(x y s' y) = \gamma_A(x y (s' y)) \leq \gamma_A(y).$$

It follows that $A = (\mu_A, \gamma_A)$ is an IFII of $S$. Similarly, we can show that $A = (\mu_A, \gamma_A)$ is an IFRI of $S$. This completes the proof.

**Theorem 3.14.** If $U$ is an interior ideal of $S$, then $\tilde{U} = (\chi_U, \tilde{\chi}_U)$ is an IFII of $S$.

**Proof.** Since $U$ is a subsemigroup of $S$, we have $\tilde{U} = (\chi_U, \tilde{\chi}_U)$ is an IFS of $S$ by Theorem 3.2. Let $x, y \in S$ be such that $x \leq y$. Then we have $x \in U$ if $y \in U$. Thus $x \leq y$ implies $\chi_U(x) = 1 = \chi_U(y)$ and $\tilde{\chi}_U(x) = 1 - \chi_U(x) = 0 = 1 - \chi_U(y) = \tilde{\chi}_U(y)$. If $y \notin U$, then $\chi_U(x) \geq \chi_U(y) = 0$ and $\tilde{\chi}_U(x) \leq \tilde{\chi}_U(y) = 1 - \chi_U(y) = 1$. Now, let $s, x, y \in S$. From the hypothesis, $x s y \in U$ if $s \in U$. In this case,

$$\chi_U(x s y) = 1 = \chi_U(s)$$

and

$$\tilde{\chi}_U(x s y) = 1 - \chi_U(x s y) = 0 = 1 - \chi_U(s) = \tilde{\chi}_U(s).$$

If $s \notin U$, then $\chi_U(s) = 0$. Thus

$$\chi_U(x s y) \geq 0 = \chi_U(s)$$

and

$$\tilde{\chi}_U(s) = 1 - \chi_U(s) = 1 \geq \tilde{\chi}_U(x s y).$$

Consequently, $\tilde{U} = (\chi_U, \tilde{\chi}_U)$ is an IFII of $S$. 

**Theorem 3.15.** Let $S$ be regular and $U$ a non-empty subset of $S$. If $\tilde{U} = (\chi_U, \tilde{\chi}_U)$ is an IFII\(_1\) or IFII\(_2\) of $S$, then $U$ is an interior ideal of $S$.

**Proof.** It is clear that $U$ is a subsemigroup of $S$ by Theorem 3.3. Suppose that $\tilde{U} = (\chi_U, \tilde{\chi}_U)$ is an IFII\(_1\) of $S$ and $x \in SU$. In this case, $x = s u t$ for some $s, t \in S, u \in U$. It follows from (II\(_1\)) that

$$\chi_U(x) = \chi_U(s u t) \geq \chi_U(u) = 1$$

Hence $\chi_U(x) = 1$, i.e. $x \in U$. Let $x \leq y$ and $y \in U$. Then

$$\chi_U(x) \geq \chi_U(y) = 1$$
Hence $\chi_U(x) = 1$, i.e. $x \in U$. Thus $U$ is an interior ideal of $S$. Now, assume that $\tilde{U} = (\chi_U, \tilde{\chi}_U)$ is an IFLI$_2$ of $S$ and $x' \in SUS$. Then $x' = s'u't'$ for some $s', t' \in S, u' \in U$. Using $(II_2)$, we obtain

$$\tilde{\chi}_U(x') = \tilde{\chi}_U(s'u't') \leq \tilde{\chi}_U(u') = 1 - \chi_U(u') = 0,$$

and so $\tilde{\chi}_U(x') = 1 - \chi_U(x') = 0$. Therefore, $\chi_U(x') = 1$, i.e. $x' \in U$. Also, let $x, y \in S$ be such that $x \leq y$ and $y \in U$. Then we have $\tilde{\chi}_U(y) \leq \tilde{\chi}_U(y)$, i.e. $1 - \chi_U(x) \leq 1 - \chi_U(y)$. Thus $\chi_U(x) \geq \chi_U(y) = 1$, i.e. $\chi_U(x) = 1$, and so $x \in U$. This completes the proof.

\[\square\]

**Definition 3.16.** $S$ is called first (resp. second) intuitionistic fuzzy left simple if IFLI$_1$ (resp. IFLI$_2$) of $S$ is constant. Also, $S$ is said to be intuitionistic fuzzy left simple if is both first and second intuitionistic fuzzy left simple, i.e. every IFLI of $S$ is constant.

**Lemma 3.17.** An ordered semigroup $S$ is left (resp. right) simple if and only if $(Sa) = S$ (resp. $(aS) = S$) for every $a \in S$.

**Theorem 3.18.** If $S$ is left simple, then $S$ is intuitionistic fuzzy left simple.

**Proof.** Let $A = (\mu_A, \gamma_A)$ be an IFLI of $S$ and $x, x' \in S$. In this case, because $S$ is left simple, there exist $s, s' \in S$ such that $x \leq sx'$ and $x' \leq s'x$. Thus, since $A = (\mu_A, \gamma_A)$ is an IFLI of $S$, we get that

$$\mu_A(x) \geq \mu_A(sx') \geq \mu_A(x'), \quad \mu_A(x') \geq \mu_A(s'x) \geq \mu_A(x)$$

and

$$\gamma_A(x) \leq \gamma_A(sx') \leq \gamma_A(x'), \quad \gamma_A(x') \leq \gamma_A(s'x) \leq \gamma_A(x)$$

Hence we have $\mu_A(x) = \mu_A(x')$ and $\gamma_A(x) = \gamma_A(x')$ for all $x, x' \in S$, that is, $A(x) = A(x')$ for all $x, x' \in S$. Consequently, $S$ is intuitionistic fuzzy left simple. This completes the proof. \[\square\]

**Theorem 3.19.** If $S$ is first or second intuitionistic fuzzy left simple, then $S$ is left simple.

**Proof.** Let $U$ be a left ideal of $S$. Suppose that $S$ is first (or second) intuitionistic fuzzy left simple. Because $\tilde{U} = (\chi_U, \tilde{\chi}_U)$ is an IFLI of $S$ by Lemma 3.8, $\tilde{U} = (\chi_U, \tilde{\chi}_U)$ is an IFLI$_1$ (and IFLI$_2$) of $S$. From the hypothesis, $\chi_U$ and $\tilde{\chi}_U$ is constant. Since $U$ is non-empty, it follows that $\chi_U = 1$ (or $\tilde{\chi}_U = 0$), where $1$ and $0$ are fuzzy sets in $S$ defined by $1(x) = 1$ and $0(x) = 0$ for all $x \in S$, respectively. Thus $x \in U$ for all $x \in S$. This completes the proof. \[\square\]

**Lemma 3.20.** An ordered semigroup $S$ is simple if and only if for every $a \in S$, we have $S = (SaS)$. 
Theorem 3.21. If \( S \) is simple, then every IFII of \( S \) is constant.

Proof. Let \( A = (\mu_A, \gamma_A) \) be an IFII of \( S \) and \( x, x' \in S \). In this case, because \( S \) is simple, there exist \( s, s', t, t' \) such that \( x \leq sx't \) and \( x' \leq s't'x \). Thus, since \( A = (\mu_A, \gamma_A) \) is an IFII of \( S \), we obtain that

\[
\mu_A(x) \geq \mu_A(sx't) \geq \mu_A(x'), \quad \mu_A(x') \geq \mu_A(s't') \geq \mu_A(x)
\]

and

\[
\gamma_A(x) \leq \gamma_A(sx't) \leq \gamma_A(x'), \quad \gamma_A(x') \leq (s't') \leq \gamma_A(x).
\]

Hence we get \( \mu_A(x) = \mu_A(x') \) and \( \gamma_A(x) = \gamma_A(x') \) for all \( x, x' \in S \). Consequently, \( A = (\mu_A, \gamma_A) \) is constant. \( \square \)

Definition 3.22. For an IFS \( A = (\mu_A, \gamma_A) \) in \( S \), consider the following axioms:

\[(IB_1) \quad x \leq y \text{ implies } \mu_A(x) \geq \mu_A(y) \text{ and } \mu_A(xsy) \geq \min\{\mu_A(x), \mu_A(y)\}, \]

\[(IB_2) \quad x \leq y \text{ implies } \gamma_A(x) \leq \gamma_A(y) \text{ and } \gamma_A(xsy) \leq \max\{\gamma_A(x), \gamma_A(y)\}\]

for all \( s, x, y \in S \). Then \( A = (\mu_A, \gamma_A) \) is called an intuitionistic fuzzy bi-ideal (briefly, IFB) of \( S \) if it satisfies \((IB_1)\) and \((IB_2)\).

Theorem 3.23. If \( S \) is left simple, then every IFB of \( S \) is an IFRI of \( S \).

Proof. Let \( A = (\mu_A, \gamma_A) \) be an IFB of \( S \) and \( x, y \in S \). In this case, from the hypothesis, there exists \( s \) such that \( y \leq sx \). Thus, because \( A = (\mu_A, \gamma_A) \) is an IFB of \( S \), we have that

\[
\mu_A(xy) \geq \mu_A(xsx) \geq \min\{\mu_A(x), \mu_A(x)\} = \mu_A(x)
\]

and

\[
\gamma_A(xy) \leq \gamma_A(xsx) \leq \max\{\gamma_A(x), \gamma_A(y)\} = \gamma_A(x).
\]

It follows that \( A = (\mu_A, \gamma_A) \) is an IFRI of \( S \). \( \square \)

Theorem 3.24. If \( U \) is a bi-ideal of \( S \), then \( \bar{U} = (\chi_U, \bar{\chi}_U) \) is an IFB of \( S \).

Proof. Since \( U \) is a subsemigroup of \( S \), we obtain that \( \bar{U} = (\chi_U, \bar{\chi}_U) \) is an IFS of \( S \) by Theorem 3.2. Let \( x, y \in S \) be such that \( x \leq y \) and \( y \in U \). Then \( x \in U \), and so \( \chi(x) = \chi(y) = 1 \) and \( \bar{\chi}_U(x) = 1 - \chi_U(x) = 0 = 1 - \chi_U(y) = \bar{\chi}_U(y) \). Let \( s, x, y \in S \). From the hypothesis, \( xsy \in U \) if \( x, y \in U \). In this case,

\[
\chi_U(xsy) = 1 = \min\{\chi_U(x), \chi_U(y)\}
\]

and

\[
\bar{\chi}_U(xsy) = 1 - \chi_U(xsy) = 0 = \max\{\bar{\chi}_U(x), \bar{\chi}_U(x)\}.
\]

If \( x \notin U \) or \( y \notin U \), then \( \chi_U(x) = 0 \) or \( \chi_U(y) = 0 \). Thus

\[
\chi_U(xsy) \geq 0 = \min\{\chi_U(x), \chi_U(y)\}
\]
and
\[
\max\{\tilde{\chi}_U(x), \tilde{\chi}_U(y)\} = \max\{1 - \chi_U(x), 1 - \chi_U(y)\} = 1 - \min\{\chi_U(x), \chi_U(y)\} \\
= 1 \geq \tilde{\chi}_U(xsy).
\]

Consequently, \(\tilde{U} = (\chi_U, \tilde{\chi}_U)\) is an IFB of \(S\).

\[\square\]

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References


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