On Stratified Domination in Prisms

Süleyman Ediz

Department of Mathematics
Institute of Sciences, Yüzüncü Yil University
65080, Van, Turkey
ediz571@gmail.com

Abstract

A graph $G$ is $2$-stratified if its vertex set is partitioned into two nonempty classes (each of which is a stratum or a color class). We color the vertices in one color class red and the other class blue. Let $S$ be a $2$-stratified graph with one fixed blue vertex $v$ specified. We say that $S$ is rooted at $v$. The $S$-domination number of a graph $G$ is the minimum number of red vertices of $G$ in a red-blue coloring of the vertices of $G$ such that for every blue vertex $v$ of $G$, there is a copy of $S$ in $G$ rooted at $v$. In this paper we get a new result that $\gamma_s(G) = 2n$ ($n \neq 2k+1$ and $n \neq 2k-1$) when $G$ is a prism $C_n \times K_2$ ($n \geq 3$) and $S$ is a $2$-stratified cycle $C_{2k+1}(k \geq 2)$ rooted at a blue vertex.

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1. Introduction

In this paper we continue the study of stratification and domination in graphs started by Chartrand et al. [1,2] and studied further in [6]. A graph $G = (V,E)$ together with a fixed partition of its vertex set $V$ into nonempty subsets is called a stratified graph. If the partition is $V = \{V_1, V_2\}$, then $G$ is a $2$-stratified graph and the set $V_1$ and $V_2$ are called the strata or color classes of $G$. We ordinarily color the vertices of $V_1$ red and the vertices of $V_2$ blue. In [13], Rashidi studied a number of problems involving stratified graphs.

Let $G = (V,E)$ be a graph with vertex set $V$ and edge set $E$, and let $S \subseteq V$. The set $S$ is a dominating set (DS) if every vertex in $V \setminus S$ is adjacent to at least one vertex of $S$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A dominating set of $G$ of cardinality of $\gamma(G)$ is called $\gamma(G)$-set. The set $S$ is a total dominating set (TDS) if every vertex in $V$ is adjacent to at least one vertex of $S$. The total
domination number of $G$, denoted by $\gamma_l(G)$, is the minimum cardinality of a total dominating set. The set $S$ is a restrained dominating set (RDS) if every vertex not in $S$ is adjacent to a vertex in $S$ and to a vertex in $V \setminus S$. The restrained domination number of $G$, denoted by $\gamma_r(G)$, is the minimum cardinality of a restrained dominating set. If $S$ is simultaneously a TDS and RDS, then $S$ is a total restrained dominating set (TRDS) of $G$. The total restrained domination number of $G$, denoted by $\gamma_{tr}(G)$, is the minimum cardinality of a total restrained dominating set. The set $S$ is a $k$-dominating set if every vertex not in $S$ is adjacent to at least $k$ vertices in $S$. The $k$-domination number of $G$, denoted by $\gamma_k(G)$, is the minimum cardinality of a $k$-dominating set. For a graph $G = (V, E)$ and a subset $S \subseteq V$, we say that a vertex $v \in V$ is total double dominated by $S$ if $|N(v) \cap S| \geq 2$. If every vertex of $V$ is total double dominated by $S$, then we call $S$ a total double dominating set (TDDS) of $G$. The total double domination number $\gamma_{2d}(G)$ is the minimum cardinality of a TDDS of $G$. The independent domination number of $G$, denoted by $i(G)$, is the minimum cardinality of a dominating set in $G$ that is independent. An independent dominating set of $G$ of cardinality $i(G)$ is called an $i(G)$-set. See [4,5,7,8,9,10,11,12] related studies about stratified domination.

More precisely, let $S$ be a 2-stratified graph with one fixed blue vertex $v$ specified. We say that $S$ is rooted at the blue vertex $v$. An $S$-coloring of a graph $G$ is defined in [2] to be a red-blue coloring of the vertices of $G$ such that every blue vertex $v$ of $G$ belongs to a copy of $S$ (not necessarily induced in $G$) rooted at $v$. The $S$-domination number of $G$ $\gamma_S(G)$ of $G$ is the minimum number of red vertices of $G$ in $S$-coloring of $G$. In [1,2], an $S$-coloring of $G$ that colors $\gamma_S(G)$ vertices red is called a $\gamma_S$-coloring of $G$. The set of red vertices in a $\gamma_S$-coloring is called a $\gamma_S$-set. If $G$ has order $n$ and $G$ has no copy of $S$, then certainly $\gamma_S(S) = n$.

For notation and graph theory terminology we follow in general [3]. A cycle on $n$ vertices is denoted by $C_n$ and a path on $n$ vertices by $P_n$. A claw is the graph $K_{1,3}$. A prism is the cartesian product $G = C_n \times K_2$, $n \geq 3$, of a cycle $C_n$ and $K_2$. Throughout this paper, our prism $G$ consist of two $n$-cycles ($v_1, v_2, \ldots, v_n, v_1$ and $u_1, u_2, \ldots, u_n, u_1$ with $v_iu_i$ an edge for all $i = 1, 2, \ldots, n$) and two $n + 2$-cycles ($v_1, u_1, v_2, \ldots, u_n, v_n, v_1$) and $(u_1, v_1, v_2, \ldots, v_n, u_n, u_1$).

Our aim is to determine the $S$-domination number of a prism when $S$ is a 2-stratified cycle $C_2$ and 2-stratified cycle $C_{2k+1}(k \geq 2)$.

2 Known Results

2.1 2-stratified claws

There are eight possible choices for a 2-stratified claw rooted at a blue vertex $v$. These graphs are shown in Fig. 1. The “claw domination” for prisms has been studied in [2].

Theorem 1 ([2]). For $n \geq 3$, let $G$ be the prism $C_n \times K_2$. Then,
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Figure 1: The distinct 2-stratified claws rooted at a blue vertex v

(a) $\gamma_{Y_1}(G) = 2 \left\lfloor \frac{n}{4} \right\rfloor$
(b) $\gamma_{Y_2}(G) = 3 \left\lfloor \frac{n}{4} \right\rfloor$
(c) $\gamma_{Y_3}(G) = n$
(d) $\gamma_{Y_4}(G) = 2 \left\lfloor \frac{n}{5} \right\rfloor$ if $n \equiv 0, 3, 4 \mod 5$, $n \equiv 2, 6 \mod 10$ or $\gamma_{Y_4}(G) = 2 \left\lfloor \frac{n}{5} \right\rfloor - 1$ if $n \equiv 1, 7 \mod 10$.
(e) $\gamma_{Y_5}(G) = 2 \left\lfloor \frac{n}{2} \right\rfloor$.
(f) $\gamma_{Y_6}(G) = 2$ if $n = 3$ or $n \equiv 2, 6 \mod 10$ or $\gamma_{Y_6}(G) = 2 \left\lfloor \frac{n}{4} \right\rfloor + i$ if $n \geq 4$ and $n \equiv i \mod 4$.
(g) $\gamma_{Y_7}(G) = 2 \left\lceil \frac{n}{2} \right\rceil$.

2.2 2-stratified $C_4$

Let $X$ be a 2-stratified $C_4$ rooted at a blue vertex $v$. The five possible choices for the graph $X$ are shown in Fig. 2. (The red vertices in Fig.2. are darkened.) The $X$-domination number of a prism when $X$ is a 2-stratified cycle $C_4$ was determined in [6].

Theorem 2 ([6]). For $n \geq 3$, let $G$ be a prism $C_n \times K_2$. Then,

(a) $\gamma_{X_1}(G) = \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor$.
(b) $\gamma_{X_1}(G) = 2n$, unless $n = 4$ in which case $\gamma_{X_1}(G) = 2$.
(c) $\gamma_{X_3}(G) = n$. 
Figure 2: The five 2-stratified $C_4$

(d) $\gamma_{X_4}(G) = \left\lceil \frac{n}{3} \right\rceil$.
(e) $\gamma_{X_5}(G) = \left\lceil \frac{4n}{3} \right\rceil$.

The relationship between the $X$-domination numbers of a prism and domination type parameters is also determined in [11]. Note that in all but one of the five possible choices for a 2-stratified $C_4$ (see Fig. 2), the red vertices form a dominating set in the graph.

**Theorem 3** ([6]). For $n \geq 3$, let $G$ be a prism $C_n \times K_2$. Then,
1. $\gamma_{X_1}(G) = \gamma(G)$.
2. $\gamma_{X_4}(G) = \gamma_2(G)$.
3. $\gamma_{X_4}(G) = \gamma_t(G) + 1$ if $n \equiv 1(\text{mod} 6)$; otherwise, $\gamma_{X_4}(G) = \gamma_t(G)$.
4. $\gamma_{X_5}(G) = \gamma^{t}_{\times 2}(G) - 1$ if $n \equiv 2(\text{mod} 6)$; otherwise $\gamma_{X_5}(G) = \gamma^{t}_{\times 2}(G)$.

3. Main Results

3.1 2-stratified $C_5$

Let $S$ be a 2-stratified $C_5$ rooted at a blue vertex $v$. The nine possible choices for the graph $S$ are shown in Fig.3. (The red vertices in Fig.3. are darkened.)

3.2 Stratification in prisms

In this section, we determine the $S$-domination number of a prism when $S$ is a 2-stratified cycle $C_5$ and a 2-stratified cycle $C_{2k+1}$.

Now we state two propositions which its’ proofs obvious. See Fig.4. and Fig.5.

**Proposition 4.** Let $G$ be the prism $C_3 \times K_2$ and $S_i$ $(i = 1, 2, ..., 9)$ be one of the 2-stratified $C_5$ rooted at a blue vertex $v$ are shown Fig.3. Then,

(a) $\gamma_{S_1}(G) = \gamma(G) = \gamma_t(G) = 2$.
(b) $\gamma_{S_2}(G) = \gamma_t(G) = \gamma_{tr}(G) = 2$.
(c) $\gamma_{S_3}(G) = \gamma_t(G) = \gamma_{tr}(G) = 2$.
(d) $\gamma_{S_4}(G) = 6$. 
Proposition 5. Let $G$ be the prism $C_5 \times K_2$ and $S_i$ ($i = 1, 2, ..., 9$) be one of the 2-stratified $C_5$ rooted at a blue vertex $v$ are shown Fig.3. Then,

(a) $\gamma_{S_1}(G) = 10$.
(b) $\gamma_{S_2}(G) = 10$.
(c) $\gamma_{S_3}(G) = 10$.
(d) $\gamma_{S_4}(G) = 10$.
(e) $\gamma_{S_5}(G) = 10$.
(f) $\gamma_{S_6}(G) = 10$.
(g) $\gamma_{S_7}(G) = \gamma_{tr} = 6$.
(h) $\gamma_{S_8}(G) = \gamma_t(G) = 6$.
(i) $\gamma_{S_9}(G) = 8$.

Notice that only $C_3 \times K_2$ and $C_5 \times K_2$ prisms contain subgraphs which isomorphic to $C_5$. Thus $\gamma_{C_5}(C_n \times K_2) = 2n, (n \neq 3, 5)$ can be written. Now we state our main result which generalize to find 2-stratified $C_{2k+1}$-domination number ($k \geq 2$) for a prism of $C_n \times K_2, (n \geq 3)$.

Theorem 6. For $n \geq 3$, let $G$ be a prism $C_n \times K_2 (n \neq 2k+1, n \neq 2k-1)$
Figure 4: 2-stratified $C_5$ domination in $C_3 \times K_2$

Figure 5: 2-stratified $C_5$ domination in $C_5 \times K_2$
and $S$ be one of a 2-stratified $C_{2k+1}$ rooted at a blue vertex $v$. Then $\gamma_S(G) = 2n$.

**Proof:** There are only three ways to get a cycle graph from a prism $C_n \times K_2$.

*Case 1:* For $i \geq 1$ and $m \geq i$, a cycle of $v_i v_{i+1} \ldots v_m u_m \ldots u_{i+1} u_i v_i$ has $2m - 2i + 2$ length which is not isomorphic to $C_{2k+1}(k \geq 2)$.

*Case 2:* A cycle of $v_1 v_2 \ldots v_n v_1$ or a cycle of $u_1 u_2 \ldots u_n u_1$ which have $n$ length and isomorphic to $C_n$. For $n \neq 2k + 1$, these two cycle graphs are not isomorphic to the cycle graph $C_{2k+1}(k \geq 2)$.

*Case 3:* A cycle of $v_1 u_1 \ldots u_n v_1$ or a cycle of $u_1 v_1 \ldots v_n u_1 u_1$ which have $n + 2$ length and isomorphic to $C_{n+2}$. For $n + 2 \neq 2k + 1 (n \neq 2k - 1)$, these two cycle graphs are not isomorphic to the cycle graph $C_{2k+1}(k \geq 2)$.

**References**


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