A Note on the Growth Estimates of Entire Functions Satisfying Second Order Linear Differential Equations

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Abstract

In this paper we investigate the comparative growth of composite entire functions which satisfy second order linear differential equations.

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1 Introduction, Definitions and Notations.

For any two transcendental entire functions $f$ and $g$ defined in the open complex plane $\mathbb{C}$, Clunie [3] proved that

$$\lim_{r \to \infty} \frac{T(r, f \circ g)}{T(r, f)} = \infty \quad \text{and} \quad \lim_{r \to \infty} \frac{T(r, f \circ g)}{T(r, g)} = \infty.$$

Singh [10] proved some comparative growth properties of $\log T(r, f \circ g)$ and $T(r, f)$. He [10] also raised the problem of investigating the comparative growth of $\log T(r, f \circ g)$ and $T(r, g)$ which he was unable to solve. However, some results on the comparative growth of $\log T(r, f \circ g)$ and $T(r, g)$ are proved in [7].
Let $f$ be an entire function defined in the open complex plane $\mathbb{C}$. Kwon [6] studied on the growth of an entire function $f$ satisfying second order linear differential equation. Later Chen [4] proved some results on the growth of solutions of second order linear differential equations with meromorphic coefficients. Chen and Yang [12] established a few theorems on the zeros and growths of entire solutions of second order linear differential equations. The purpose of this paper is to study on the growth of the solution $f \neq 0$ of the second order linear differential equation

$$f'' + A(z)f' + B(z)f = 0,$$

where $A(z)$ and $B(z) \neq 0$ are entire functions. We do not explain the standard notations and definitions in the theory of entire and meromorphic functions because those are available in [11] and [5].

The following definitions are well known.

**Definition 1.** The order $\rho_f$ and lower order $\lambda_f$ of an entire function $f$ is defined as

$$\rho_f = \limsup_{r \to \infty} \frac{\log^{[2]} M(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^{[2]} M(r, f)}{\log r},$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, \ldots$ and $\log^{[0]} x = x$.

If $f$ is meromorphic, one can easily verify that

$$\rho_f = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r}.$$ 

**Definition 2.** The hyper order $\tilde{\rho}_f$ and hyper lower order $\tilde{\lambda}_f$ of an entire function $f$ is defined as follows

$$\tilde{\rho}_f = \limsup_{r \to \infty} \frac{\log^{[3]} M(r, f)}{\log r} \quad \text{and} \quad \tilde{\lambda}_f = \liminf_{r \to \infty} \frac{\log^{[3]} M(r, f)}{\log r}.$$

If $f$ is meromorphic, then

$$\tilde{\rho}_f = \limsup_{r \to \infty} \frac{\log^{[2]} T(r, f)}{\log r} \quad \text{and} \quad \tilde{\lambda}_f = \liminf_{r \to \infty} \frac{\log^{[2]} T(r, f)}{\log r}.$$ 

**Definition 3.** [8] Let $f$ be an entire function of order zero. Then the quantities $\rho^*_f$, $\lambda^*_f$ and $\tilde{\rho}^*_f$, $\tilde{\lambda}^*_f$ are defined in the following way:

$$\rho^*_f = \limsup_{r \to \infty} \frac{\log^{[2]} M(r, f)}{\log^{[2]} r}, \quad \lambda^*_f = \liminf_{r \to \infty} \frac{\log^{[2]} M(r, f)}{\log^{[2]} r}.$$
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and

\[ \rho_f^* = \limsup_{r \to \infty} \frac{\log^3 M(r, f)}{\log^2 r}, \quad \lambda_f^* = \liminf_{r \to \infty} \frac{\log^3 M(r, f)}{\log^2 r} . \]

If \( f \) is meromorphic then clearly

\[ \rho_f^* = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log^2 r}, \quad \lambda_f^* = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log^2 r} . \]

and

\[ \rho_f^* = \limsup_{r \to \infty} \frac{\log^2 T(r, f)}{\log^2 r}, \quad \lambda_f^* = \liminf_{r \to \infty} \frac{\log^2 T(r, f)}{\log^2 r}. \]

**Definition 4.** The type \( \sigma_f \) of an entire function \( f \) is defined as

\[ \sigma_f = \limsup_{r \to \infty} \frac{\log M(r, f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty. \]

When \( f \) is meromorphic, then

\[ \sigma_f = \limsup_{r \to \infty} \frac{T(r, f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty. \]

**Definition 5.** Let \( a \) be a complex number, finite or infinite. The Nevanlinna deficiency and the Valiron deficiency of \( a \) with respect to a meromorphic function \( f \) are defined as

\[ \delta(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r \to \infty} \frac{m(r, a; f)}{T(r, f)} \]

and

\[ \Delta(a; f) = 1 - \liminf_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} = \limsup_{r \to \infty} \frac{m(r, a; f)}{T(r, f)}. \]

**2 Lemmas.**

In this section we present some lemmas which will be needed in the sequel.

**Lemma 1.** [1] If \( f \) is meromorphic and \( g \) is entire then for all sufficiently large values of \( r \),

\[ T(r, f \circ g) \leq \{1 + o(1)\} \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f). \]
Lemma 2. [2] Let $f$ be meromorphic and $g$ be entire and suppose that $0 < \mu \leq \rho_g \leq \infty$. Then for a sequence of values of $r$ tending to infinity,

$$T(r, f \circ g) \geq T(\exp(r^\mu), f).$$

Lemma 3. [9] Let $f$ and $g$ be two transcendental entire functions with $\rho_g < \infty$, $\eta$ be a constant satisfying $0 < \eta < 1$ and $\alpha$ be a positive number. Then

$$T(r, f \circ g) + O(1) \geq N(r, 0; f \circ g) \geq \log \left( \frac{1}{\eta} \right) \left[ \frac{N(M((\eta r)^{1/\alpha}, g), 0, f) - O(1)}{\log M((\eta r)^{1/\alpha}, g)} \right]$$

as $r \to \infty$ through all values.

3 Theorems.

In this section we present the main results of the paper.

Theorem 1. Let $f$ be an entire function satisfying the second order linear differential equation $f'' + A(z)f' + B(z)f = 0$ where $A(z)$ and $B(z) \not\equiv 0$ are entire functions. If (i) $\rho_A, \rho_B$ are both finite, (ii) $\lambda_A, \lambda_f$ are both positive, (iii) $\rho_B < \lambda_A$ and $\rho_B < \lambda_f$ i.e. $\rho_B < \min \{\lambda_A, \lambda_f\}$ and (iv) $B$ be of regular growth i.e., $\lambda_B = \rho_B$ then

$$\lim_{r \to \infty} \frac{\log T(r, A \circ B)}{T(r, f)} = 0.$$

Proof. It is well known that for an entire function $B$, $T(r, B) \leq \log^+ M(r, B)$. So in view of Lemma 1, we get for all sufficiently large values of $r$,

$$T(r, A \circ B) \leq \{1 + o(1)\} T(M(r, B), A)$$

i.e.,

$$\log T(r, A \circ B) \leq \log \{1 + o(1)\} + \log T(M(r, B), A)$$

i.e.,

$$\log T(r, A \circ B) \leq o(1) + (\rho_A + \epsilon) \log M(r, B)$$

i.e.,

$$\log T(r, A \circ B) \leq o(1) + (\rho_A + \epsilon)r^{\rho_B + \epsilon}.$$

(1)

Also we obtain for all sufficiently large values of $r$,

$$T(r, A) \geq r^{\lambda_A - \epsilon}.$$

(2)

Now combining (1) and (2) it follows for all sufficiently large values of $r$,

$$\frac{\log T(r, A \circ B)}{T(r, A)} \leq \frac{o(1) + (\rho_A + \epsilon)r^{(\rho_B + \epsilon)}}{r^{\lambda_A - \epsilon}}$$

i.e.,

$$\limsup_{r \to \infty} \frac{\log T(r, A \circ B)}{T(r, A)} \leq \limsup_{r \to \infty} \frac{o(1) + (\rho_A + \epsilon)r^{(\rho_B + \epsilon)}}{r^{\lambda_A - \epsilon}}.$$
Since $\rho_B < \lambda_A$, we can choose $\epsilon (> 0)$ in such a way that $\rho_B + \epsilon < \lambda_A - \epsilon$ and so it follows from above that

$$\lim_{r \to \infty} \frac{\log T(r, A \circ B)}{T(r, A)} = 0.$$  \hspace{1cm} (3)

Again we get for all sufficiently large values of $r$,

$$\log T(r, f) \geq (\lambda_f - \epsilon) \log r$$

i.e., $T(r, f) \geq r^{\lambda_f - \epsilon}$.

$$\text{i.e., } \lim_{r \to \infty} \frac{\log T(r, A \circ B)}{T(r, f)} = 0.$$ \hspace{1cm} (4)

Since $\rho_B < \lambda_f$, we can choose $\epsilon (> 0)$ in such a way that

$$\rho_B + \epsilon < \lambda_f - \epsilon.$$ \hspace{1cm} (5)

Now combining (1), (4) and (5) it follows for all sufficiently large values of $r$,

$$\frac{\log T(r, A \circ B)}{T(r, f)} \leq \frac{o(1) + (\rho_A + \epsilon) r^{(\rho_B + \epsilon)}}{r^{\lambda_f - \epsilon}}$$

i.e., $\lim_{r \to \infty} \frac{\log T(r, A \circ B)}{T(r, f)} = 0.$ \hspace{1cm} (6)

Therefore in view of (3) and (6), we obtain that

$$\lim_{r \to \infty} \frac{\{\log T(r, A \circ B)\}^2}{T(r, f)T(r, B)} = 0.$$ 

i.e., $\lim_{r \to \infty} \frac{\{\log T(r, A \circ B)\}^2}{T(r, f)T(r, B)} = 0.$

This proves the theorem.

**Theorem 2.** Let $f$ be an entire function satisfying the second order linear differential equation $f'' + A(z)f' + B(z)f = 0$ where $A(z)$ and $B(z) \not\equiv 0$ are entire functions. If $\rho_B = 0$ then $\rho_{A \circ B} \geq \lambda_A^* \mu$ where $0 < \mu < \rho_B$. 
Proof. In view of Lemma 2 and for $0 < \mu < \rho_B$ we get that

\[
\rho_{A \circ B} = \limsup_{r \to \infty} \frac{\log T(r, A \circ B)}{\log r} \\
\geq \liminf_{r \to \infty} \frac{\log T(\exp(r\mu), A)}{\log r} \\
\geq \liminf_{r \to \infty} \frac{\log T(\exp(r\mu), A)}{\log |\exp(r\mu)|} \liminf_{r \to \infty} \frac{\log^2(\exp(r\mu))}{\log r} \\
= \lambda_A^* \liminf_{r \to \infty} \frac{\log r}{\log r} \\
= \lambda_A^* \mu.
\]

Thus the theorem is established.

Theorem 3. Let $f$ be an entire function satisfying the second order linear differential equation $f'' + A(z)f' + B(z)f = 0$ where $A(z)$ and $B(z) \neq 0$ are entire functions. If $\rho_A, \rho_B$ are both finite and $\lambda_f$ is positive then for any $\alpha \in (-\infty, \infty)$,

\[
\lim_{r \to \infty} \frac{\log \{T(r, A \circ B) \log M(r, B)\}}{T(\exp r, f)}^{1+\alpha} = 0.
\]

Proof. If $1 + \alpha \leq 0$, the theorem is obvious. So we suppose that $1 + \alpha > 0$. In view of Lemma 1, we have for all sufficiently large values of $r$,

\[
\log \{T(r, A \circ B) \log M(r, B)\} \\
\leq \log T(r, B) + \log T(M(r, B), A) + \log\{1 + o(1)\} \\
\leq (\rho_B + \epsilon) \log r + (\rho_A + \epsilon) r^{\rho_B + \epsilon} + o(1) \\
\leq r^{\rho_B + \epsilon} \left\{ (\rho_A + \epsilon) + \frac{(\rho_B + \epsilon) \log r + o(1)}{r^{\rho_B + \epsilon}} \right\}. \quad (7)
\]

Again we get for all sufficiently large values of $r$,

\[
\log T(\exp r, f) \geq (\lambda_f - \epsilon) \log \{\exp r\} \quad \text{i.e.,} \quad T(\exp r, f) \geq \exp \{(\lambda_f - \epsilon) r\}. \quad (8)
\]

Now combining (7) and (8) it follows for all sufficiently large values of $r$,

\[
\frac{\log \{T(r, A \circ B) \log M(r, B)\}}{T(\exp r, f)}^{1+\alpha} \\
\leq \frac{r^{(\rho_B + \epsilon)(1+\alpha)} \left\{ (\rho_A + \epsilon) + \frac{(\rho_B + \epsilon) \log r + o(1)}{r^{\rho_B + \epsilon}} \right\}^{1+\alpha}}{\exp \{(\lambda_f - \epsilon) r\}}.
\]
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i.e., \[ \limsup_{r \to \infty} \frac{\log \{ T(r, A \circ B) \log M(r, B) \}^{1+\alpha}}{T(\exp(r), f)} = 0, \]
from which the theorem follows.

**Theorem 4.** Let \( f \) be an entire function satisfying the second order linear differential equation \( f'' + A(z)f' + B(z)f = 0 \) where \( A(z) \) and \( B(z) \neq 0 \) are entire functions. If \( \rho_A, \rho_B \) are both finite and \( \lambda_f \) is positive then for any \( \alpha \in (-\infty, \infty) \),

\[
\lim_{r \to \infty} \frac{\log \{ T(r, A \circ B) \log M(r, B) \}^{1+\alpha}}{T(\exp r, f)} = 0 \text{ if } 0 < 1 + \alpha < \frac{1}{\rho_B}.
\]

**Proof.** If \( 1 + \alpha \leq 0 \), the theorem is obvious. So we take \( 1 + \alpha > 0 \).

We obtain for all sufficiently large values of \( r \),

\[
\log T(\exp r, f) \geq (\lambda_f - \epsilon) \log \{ \exp r \}
\]

i.e., \( \log T(\exp r, f) \geq (\lambda_f - \epsilon)r. \) \hspace{1cm} (9)

Now from (7) and (9) it follows for sufficiently large values of \( r \),

\[
\frac{\log \{ T(r, A \circ B) \log M(r, B) \}^{1+\alpha}}{\log T(\exp r, f)} \leq \frac{r^{(\rho_B + \epsilon)(1+\alpha)} \{ (\rho_A + \epsilon) + \rho_B + \epsilon(1) \}^{1+\alpha}}{(\lambda_f - \epsilon)r^\alpha}. \hspace{1cm} (10)
\]

Since \( 1 + \alpha < \frac{1}{\rho_B} \), we can choose \( \epsilon (>0) \) in such a way that

\[
(\rho_B + \epsilon)(1 + \alpha) < 1. \hspace{1cm} (11)
\]

Thus the theorem follows from (10) and (11).

**Theorem 5.** If \( f \) be an entire function satisfying the second order linear differential equation \( f'' + A(z)f' + B(z)f = 0 \) where \( A(z) \) and \( B(z) \neq 0 \) are entire functions. If \( 0 < \lambda_{A \circ B} \leq \rho_{A \circ B} < \infty \) and \( 0 < \rho_f < \infty \) then for any positive number \( \alpha \),

\[
\liminf_{r \to \infty} \frac{\log^2 T(r, A \circ B)}{\log^2 T(r^\alpha, f)} \leq \frac{\rho_{A \circ B}}{\alpha \rho_f} \leq \limsup_{r \to \infty} \frac{\log^2 T(r, A \circ B)}{\log^2 T(r^\alpha, f)}.
\]

**Proof.** From the definition of hyper order we get for all sufficiently large values of \( r \),

\[
\log^2 T(r, A \circ B) \leq (\rho_{A \circ B} + \epsilon) \log r. \hspace{1cm} (12)
\]
Again we have for a sequence of values of $r$ tending to infinity,

$$\log^2 T(r^\alpha, f) \geq (\bar{\rho}_f - \epsilon) \log r^\alpha$$

i.e.,

$$\log^2 T(r^\alpha, f) \geq \alpha(\bar{\rho}_f - \epsilon) \log r.$$  \hfill (13)

Now combining (11) and (12) it follows for a sequence of values of $r$ tending to infinity,

$$\frac{\log^2 T(r, A \circ B)}{\log^2 T(r^\alpha, f)} \leq \frac{(\bar{\rho}_{A \circ B} + \epsilon) \log r}{\alpha (\bar{\rho}_f - \epsilon) \log r}.$$ \hfill (14)

Since $\epsilon (> 0)$ is arbitrary, it follows from above that

$$\liminf_{r \to \infty} \frac{\log^2 T(r, A \circ B)}{\log^2 T(r^\alpha, f)} \leq \frac{\bar{\rho}_{A \circ B}}{\alpha \bar{\rho}_f}.$$ \hfill (15)

Also for arbitrary positive $\epsilon$ and for all sufficiently large values of $r$,

$$\log^2 T(r^\alpha, f) \leq (\bar{\rho}_f + \epsilon) \log r^\alpha$$

i.e.,

$$\log^2 T(r^\alpha, f) \leq \alpha(\bar{\rho}_f + \epsilon) \log r.$$ \hfill (16)

Again for a sequence of values of $r$ tending to infinity,

$$\frac{\log^2 T(r, A \circ B)}{\log^2 T(r^\alpha, f)} \geq \frac{(\bar{\rho}_{A \circ B} - \epsilon) \log r}{\alpha (\bar{\rho}_f + \epsilon) \log r}.$$ \hfill (17)

As $\epsilon (> 0)$ is arbitrary, we have from above that

$$\limsup_{r \to \infty} \frac{\log^2 T(r, A \circ B)}{\log^2 T(r^\alpha, f)} \geq \frac{\bar{\rho}_{A \circ B}}{\alpha \bar{\rho}_f}.$$ \hfill (18)

Thus the theorem follows from (13) and (16).

**Theorem 6.** Let $f$ be an entire function satisfying the second order linear differential equation $f'' + A(z)f' + B(z)f = 0$ where $A(z)$ and $B(z)$ $\not\equiv 0$ are entire functions. If (i) $0 < \rho_f < \infty$, (ii) $\sigma_f < \infty$, (iii) $\rho_{A \circ B} = \rho_f$ and (iv) $0 < \sigma_{A \circ B} < \infty$ then

$$\liminf_{r \to \infty} \frac{T(r, A \circ B)}{T(r, f)} \leq \frac{\sigma_{A \circ B}}{\sigma_f} \leq \limsup_{r \to \infty} \frac{T(r, A \circ B)}{T(r, f)}.$$
Proof. By the definition of type, we have for arbitrary positive \( \epsilon \) and for all sufficiently large values of \( r \),

\[
T(r, A \circ B) \leq (\sigma_{A \circ B} + \epsilon)r^{\rho_{A \circ B}}.
\]  
(18)

Again we get for a sequence of values of \( r \) tending to infinity,

\[
T(r, f) \geq (\sigma_f - \epsilon)r^{\rho_f}.
\]  
(19)

Since \( \rho_{A \circ B} = \rho_f \) from (17) and (18) it follows for a sequence of values of \( r \) tending to infinity,

\[
\frac{T(r, A \circ B)}{T(r, f)} \leq \frac{(\sigma_{A \circ B} + \epsilon)}{(\sigma_f - \epsilon)}.
\]  
(20)

As \( \epsilon \ (> 0) \) is arbitrary, it follows from above that

\[
\liminf_{r \to \infty} \frac{T(r, A \circ B)}{T(r, f)} \leq \frac{\sigma_{A \circ B}}{\sigma_f}.
\]  
(21)

Again for a sequence of values of \( r \) tending to infinity,

\[
\frac{T(r, A \circ B)}{T(r, f)} \geq \frac{(\sigma_{A \circ B} - \epsilon)}{(\sigma_f + \epsilon)}.
\]  
(22)

Now in view of condition (iii) we get from (20) and (21) for a sequence of values of \( r \) tending to infinity,

\[
\limsup_{r \to \infty} \frac{T(r, A \circ B)}{T(r, f)} \geq \frac{\sigma_{A \circ B}}{\sigma_f}.
\]  
(23)

Thus the theorem follows from (19) and (22).

**Theorem 7.** Let \( f \) be an entire function satisfying the second order linear differential equation \( f'' + A(z)f' + B(z)f = 0 \) where \( A(z) \) and \( B(z) \) \( \neq 0 \) are entire functions. If (i) \( 0 < \lambda_B \leq \rho_B < \infty \), (ii) \( \lambda_A > 0 \), (iii) \( \rho_f < \infty \) and (iv) \( \Delta(0; A) < 1 \) then

\[
\lim_{r \to \infty} \frac{\log T(r, A \circ B)}{\log T(r^\beta, f)} = \infty,
\]

where \( \beta \) is a real constant.
Proof. We suppose that $\beta > 0$ because otherwise the theorem is obvious. For given $\epsilon$ ($0 < \epsilon < 1 - \Delta(0; A)$),

$$N(r, 0; A) > (1 - \Delta(0; A) - \epsilon)T(r, A)$$

for all sufficiently large values of $r$.

So from Lemma 3 we get for all large values of $r$,

$$T(r, A \circ B) + O(1) \geq \left(\log \frac{1}{\eta} \right) \left[ (1 - \Delta(0; A) - \epsilon)T \left\{ M((\eta r)^{1+\alpha}, B), A \right\} \right] - O(1).$$

(24)

Since for all large values of $r$, $\log M(r, B) < r^{\rho_B + \epsilon}$, it follows from (23) that for all sufficiently large values of $r$,

$$T(r, A \circ B) + O(1) \geq O(\log r) + \log T \left\{ M((\eta r)^{1+\alpha}, B), A \right\} + \log \left[ 1 - \frac{\log M((\eta r)^{1+\alpha}, B)O(1)}{(1 - \Delta(0; A) - \epsilon)T \left\{ M((\eta r)^{1+\alpha}, B), A \right\}} \right].$$

Since $f$ is transcendental, it follows that

$$\lim_{r \to \infty} \frac{\log M((\eta r)^{1+\alpha}, B)}{T \left\{ M((\eta r)^{1+\alpha}, B), A \right\}} = 0.$$

So from above we get for all large values of $r$,

$$\log T(r, A \circ B) \geq O(\log r) + \log T \left\{ M((\eta r)^{1+\alpha}, B), A \right\} + o(1).$$

(25)

Also we see that for all large values of $r$,

$$M(r, B) > \exp \left\{ (r)^{(1/2)\lambda_B} \right\}$$

$$\log T(r, A) > \frac{1}{2}\lambda_A \log r$$

and

$$T(r, f) < r^{\rho_f + 1}.$$

So from (24) we obtain for all sufficiently large values of $r$,

$$\frac{\log T(r, A \circ B)}{\log T(r^\beta, f)} > \frac{O(\log r)}{\beta(1 + \rho_B) \log r} + \frac{\lambda_A}{2} \cdot \frac{(\eta r)^{\frac{\lambda_B}{\beta(1 + \rho_B) \log r}}}{\beta(1 + \rho_B) \log r} + o(1),$$

which implies that

$$\lim_{r \to \infty} \frac{\log T(r, A \circ B)}{\log T(r^\beta, f)} = \infty.$$
References


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