Common Fixed Point Theorem in M-Fuzzy Metric Spaces Using Implicit Relation

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Abstract

The purpose of this paper is to improve and extend the results of Bijendra Singh and Shishir Jain [3] and Popa [14,15] by extending the number of weak compatible mappings from four to six and use different implicit relation and replacing fuzzy metric space to M-Fuzzy metric space.

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1 Introduction

After introduction of fuzzy sets by Zadeh [6], Kramosil and Michalek [5] introduced the concept of fuzzy metric space in 1975. Consequently in due course of time many researchers have defined fuzzy metric space in different ways. Researchers like George and Veeramani [1], Grabiec [7], Subrahmanyam [9], Vasuki [11] used this concept to generalize some metric fixed point results. Recently Sedghi and Shobe [12] introduced M-fuzzy metric space which is based on D*-metric concept. Popa [14] proved theorem for weakly compatible non-continuous mappings using implicit relation. Chauhan [13] proved some results using four weak compatible mappings in M-fuzzy metric space. The main object of this paper is to obtain some common fixed
point theorems in $M$-fuzzy metric space using “Implicit Relation”, in which we generalize and improve the result of [3], [13] and [14, 15] by
(i) replacing the fuzzy metric space by $M$-fuzzy metric space.
(ii) increase the number of self maps from four to six.
First we give some known definitions and results in $M$-fuzzy metric space given by Sedghi and Shobe [12] and then prove our main result.

**Definition 1.1** ([2]) A binary operation $*: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norm if it satisfies the following conditions
1. $*$ is associative and commutative,
2. $*$ is continuous,
3. $a*1 = a$ for all $a \in [0, 1],$
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1].$
Two typical examples of continuous t-norm are $a*b = a b$ and $a*b = \min \{a, b\}.$

**Definition 1.2** ([12]) A 3-tuple $(X, M, *)$ is called a $M$-fuzzy metric space if $X$ is an arbitrary (non-empty) set, $*$ is a continuous t-norm, and $M$ is a fuzzy set on $X^3 \times (0, \infty),$ satisfying the following conditions for each $x, y, z, a \in X$ and $t, s > 0$,
1. $M(x, y, z, t) > 0,$
2. $M(x, y, z, t) = 1$ if and only if $x = y = z,$
3. $M(x, y, z, t) = M(p\{x, y, z\}, t),$ (symmetry) where $p$ is a permutation function,
4. $M(x, y, a, t) * M(a, z, z, s) \leq M(x, y, z, t + s),$
5. $M(x, y, z, .): (0, \infty) \rightarrow [0, 1]$ is continuous.

**Remark 1.1** ([12]) Let $(X, M, *)$ be a $M$-fuzzy metric space. Then for every $t > 0$ and for every $x, y \in X$ we have $M(x, x, y, t) = M(x, y, y, t).$

**Definition 1.3** ([12]) Let $(X, M, *)$ be a $M$-fuzzy metric space. For $t > 0,$ the open ball $BM(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by $BM(x, r, t) = \{y \in X: M(x, y, y, t) > 1 - r\}.$
A subset $A$ of $X$ is called open set if for each $x \in A$ there exist $t > 0$ and $0 < r < 1$ such that $BM(x, r, t) \subseteq A.$

**Definition 1.4** ([12]) A sequence $\{x_n\}$ in $X$ converges to $x$ if and only if $M(x, x, x_n, t) \rightarrow 1$ as $n \rightarrow \infty,$ for each $t > 0$. It is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0,$ there exists $n_0 \in \mathbb{N}$ such that $M(x, x_m, x_n, t) > 1 - \varepsilon$ for each $n, m \geq n_0.$
The $M$-fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent.

**Lemma 1.1** ([12]). Let $(X, M, *)$ be a $M$-fuzzy metric space. Then $M(x, y, z, t)$ is non-decreasing with respect to $t,$ for all $x, y, z$ in $X.$
Lemma 1.2 ([12]). Let \((X, M, *)\) be a \(M\)-fuzzy metric space. Then \(M\) is continuous function on \(X^3 \times (0, \infty)\).


Definition 1.5 ([2]) Let \(f\) and \(g\) be two self maps of \((X, M, *)\). Then \(f\) and \(g\) are said to be weakly compatible if there exists \(u\) in \(X\) with \(fu = gu\) implies \(fgu = gfu\).

Here we introduce new implicit relation and an example in support of our main result.

A class of implicit relation. Let \(\Phi\) be the set of all real continuous functions \(\phi: (\mathbb{R}^+)^4 \to \mathbb{R}\), nondecreasing and satisfying the following conditions.

(i) For \(u, v \geq 0\), \(\phi(v, v, u, u) \geq 0\) implies that \(u \geq v\).

(ii) \(\phi(u, 1, 1, u) \geq 0\) implies that \(u \geq 1\).

(iii) \(\phi(1, 1, u, u) \geq 0\) implies that \(u \geq 1\).

(iv) \(\phi(u, 1, u, u) \geq 0\) implies that \(u \geq 1\).

Example. Define \(\phi(t_1, t_2, t_3, t_4) = 5t_1 - 6t_2 + 2t_3 - t_4\). Then \(\phi \in \Phi\).

2. Main Results

Theorem 2.1. Let \(A, B, S, T, I\) and \(J\) be self maps of a complete \(M\)-fuzzy metric space \((X, M, *)\) with continuous \(t\)-norm \(*\) defined by \(a * b = \min (a, b), a, b \in [0, 1]\) satisfying

(I) \(AB(X) \subseteq J(X), ST(X) \subseteq I(X)\)

(II) \((AB, J)\) and \((ST, I)\) are weakly compatible. For some \(\phi \in \Phi, x, y, z \in X, t > 0,\)

(III) \(\phi\{ M(ABx, S Ty, Iz, t), M(ABy, Jx, STy, t), M(STz, Iz, Jx, t), M(ABx, STz, Iz, t) \} \geq 0\).

Then \(AB, ST, I\) and \(J\) have unique common fixed point.

Proof. Let \(x_0 \in X\) be any arbitrary point. Since \(AB(X) \subseteq J(X)\) and \(ST(X) \subseteq I(X)\), there exist a point \(x_1, x_2 \in X\) such that \(ABx_0 = Jx_1\) and \(STx_1 = Ix_2\). Inductively we get a sequence \(\{y_n\}\) as \(y_{2n-1} = Jx_{2n-1} = ABx_{2n-2}, y_{2n} = Ix_{2n} = STx_{2n-1}, n = 1, 2, \ldots\).

Let \(M_n = M(y_n, y_{n+1}, y_{n+2}, t) < 1\) for all \(n\)

Put \(x = x_{2n+1}, y = x_{2n-2}, z = x_{2n}\) in (III), we get

\(\phi\{ M(ABx_{2n+1}, STx_{2n-2}, Ix_{2n}, t), M(ABx_{2n-2}, Jx_{2n+1}, STx_{2n-2}, t), M(STx_{2n}, Ix_{2n}, Jx_{2n+1}, t), M(ABx_{2n+1}, STx_{2n}, Ix_{2n}, t) \} \geq 0\) implies

\(\phi\{ M(y_{2n+1}, y_{2n-1}, y_{2n}, t), M(y_{2n-1}, y_{2n+1}, y_{2n}, t), M(y_{2n+1}, y_{2n}, y_{2n+1}, t), M(y_{2n+2}, y_{2n+1}, y_{2n}, t) \} \geq 0\)

implies \(\phi\{ M_{2n+1}, M_{2n-1}, M_{2n}, M_{2n} \} \geq 0\) thus, we have \(M_{2n} \geq M_{2n-1}\) (1)

Thus \(\{M_{2n}, n \geq 0\}\) is an increasing sequence of positive real numbers in \([0, 1]\) and therefore tends to limit \(m \leq 1\).
We claim \( m = 1 \), for \( m < 1 \), taking limit in (1), we get \( m < m \), which is a contradiction. Therefore \( m = 1 \).

For any \( + \) ve integer \( r \),

\[
M(y_n, y_n, y_{n+r}, t) \geq M(y_{n+1}, y_{n+1}, y_{n+2}, \frac{t}{r}) * \ldots * M(y_{n+r-1}, y_{n+r-1}, y_{n+r}, \frac{t}{r}) > (1 - \varepsilon)^r
\]

implies \( M(y_n, y_n, y_{n+r}, t) > (1 - \varepsilon) \)

Thus, \( M(y_n, y_n, y_{n+s}, t) > (1 - \varepsilon) \) for \( n, s \geq n_0 \) where \( n_0 \in \mathbb{N} \).

Thus, \( \{y_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete there is a point \( p \in X \) s.t. \( y_n \to p \)

Thus, subsequences \( \{ABx_{2n}\}, \{Jx_{2n-1}\}, \{Ix_{2n}\} \) also converges to \( p \).

Since \( AB(X) \subset J(X) \) and \( ST(X) \subset I(X) \) then there must exist \( u, v \in X \) s.t. \( p = Jv, p = Iu \).

Put \( x = v, y = x_{2n-1}, z = x_{2n} \) in condition (III)

\[
\phi\{M(ABv, STx_{2n-1}, Ix_{2n}, t), M(ABx_{2n-1}, Jv, STx_{2n-1}, Ix_{2n}, t), M(ABv, STx_{2n-1}, Ix_{2n}, t)\} \geq 0
\]

which implies

\[
\phi\{M(p, p, p, t), M(p, p, p, t), M(p, p, p, t)\} \geq 0
\]

implies \( \phi\{1, 1, M(p, p, p, t)\} \geq 0 \)

Thus, \( STu = p \).

Hence, \( ABv = p = Jv \).

Put \( x = v, y = x_{2n+1}, z = u \) in (III), we get

\[
\phi\{M(ABv, STx_{2n+1}, Iu, t), M(ABx_{2n+1}, Jv, STx_{2n+1}, Iu, t), M(ABv, STu, Iu, t)\} \geq 0
\]

which implies

\[
\phi\{M(p, p, p, t), M(p, p, p, t), M(p, p, p, t)\} \geq 0
\]

implies \( M(p, p, p, t) \geq 1 \),

Therefore, \( STu = p \).

Hence, \( ABv = p \). Consequently \( ABv = Jv = Iu = STu \).

Since, \( (AB, J) \) and \( (ST, I) \) are weak compatible, therefore \( ABJv = JABv \) implies \( ABp = Jp \).

Therefore, \( p \) is a coincident point of \( AB, ST, I, \) and \( J \).

Now, we shall prove that \( p \) is a fixed point of \( AB, ST, I \) and \( J \)

Put \( x = v, y = x_{2n+1}, z = p \) in (III), we get

\[
\phi\{M(ABv, STx_{2n+1}, Ip, t), M(ABx_{2n+1}, Jv, STx_{2n+1}, Ip, t), M(STp, Ip, Jv, t), M(ABv, STp, Ip, t)\} \geq 0
\]

\[
\phi\{M(p, p, Ip, t), M(p, p, Ip, t), M(p, Ip, Ip, t)\} \geq 0
\]

implies \( M(p, p, Ip, t) \geq 1 \),

Therefore, \( Ip = p \).

Hence, \( STp = p \).

Similarly, \( ABp = Jp = p \).

Therefore, \( ABp = Jp = Ip = STp = p \)

Consequently, \( p \) is a common fixed point of \( AB, ST, I \) and \( J \).
**UNIQUENESS**

Let \( w \) be a fixed point other than \( p \) of \( AB, ST, I \) and \( J \).
Then, \( ABw = STw = Iw = Jw = w \)
Put \( x = p, y = p, z = w \) in (III), we get
\[
\varphi \{M(ABp, STp, Iw, t), M(ABp, Jp, STp, t), M(STw, Iw, Jp, t), M(ABp, STw, Iw, t)\} \geq 0 ,
\]
which implies
\[
\varphi \{M(p, p, w, t), M(p, p, p, t), M(w, w, p, t), M(p, w, w, t)\} \geq 0 ,
\]
implies
\[
\varphi \{M(p, p, w, t), M(p, p, w, t)\} \geq 0 ,
\]
\( M(p, p, w, t) \geq 1 \)
Therefore \( p = w \)
Hence, \( p \) is a unique fixed point of \( AB, ST, I \) and \( J \).

**Corollary 2.1.** Let \( A, S, I \) and \( J \) be self maps of a complete \( M \)-fuzzy metric space \((X, M, *)\) with continuous \( t \)-norm \( * \) defined by \( a * b = \min (a, b) \), \( a, b \in [0, 1] \) satisfying

(I) \( A(X) \subset J(X) \), \( S(X) \subset I(X) \)
(II) \( A, J \) and \( S, I \) are weakly compatible.
For some \( \varphi \in \Phi, x,y,z \in X, t > 0, \)
(III) \( \varphi \{M(Ax, Sy, Iz, t), M(Ay, Jx, Sy, t), M(Sz, Ix, Jx, t), M(Ax, Sz, Iz, t)\} \geq 0 \)
Then \( A, S, I \) and \( J \) have unique common fixed point.
Proof. Taking \( B = T = Id = Identity mapping \) in above theorem, we get the required result.

**References**


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