Some New Types of Open and Closed Sets in Minimal Structures-I

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Abstract. In this article different forms of closed sets in $m$-spaces are introduced and studied. We show that the obtained results are a generalization of many of the results obtained by G. Aslim et al. in [1].

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1. Introduction

In the literature, the notions of $g$-closed set, gs-closed set, sg-closed set, $\pi$-closed set, $\pi g$-closed set, $\pi$gs-closed set, gp-closed set, $\pi$gp-closed set, and their relationships, are studied in a topological space as well as the different notions of continuous functions and irresolute functions, where they use the concepts mentioned previously. The fundamental idea of this article is to define

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the above notions on a \( m \)-space [8]. Also, we look for conditions on the \( m \)-structure in order to generalize the well known results in this matter. Moreover, we find the existent relation between the different notions of continuity and irresoluteness and also we find conditions under what the direct image of any \( m_X \)-sg-open set in \( X \) is \( m_Y \)-sg-open in \( Y \) and the inverse image of any \( m_Y \)-sg-open set in \( Y \) is \( m_X \)-sg-open in \( X \). Finally, we show that the obtained results are a generalization of many of the results obtained by G. Aslim et al. in [1].

2. Minimal Structures

In this section, we introduce the \( m \)-structure and define some important subsets associated to the \( m \)-structure and the relation between them.

**Definition 2.1.** Let \( X \) be a nonempty set and let \( m_X \subseteq P(X) \), where \( P(X) \) denote the set of power of \( X \). We say that \( m_X \) is an \( m \)-structure (or a minimal structure) on \( X \), if \( \emptyset \) and \( X \) belong to \( m_X \).

The members of the minimal structure \( m_X \) are called \( m_X \)-open sets, and the pair \((X,m_X)\) is called an \( m \)-space. The complement of an \( m_X \)-open set is called \( m_X \)-closed. Given \( A \subseteq X \), we define \( m_X \)-interior of \( A \) abbreviate \( m_X \)-Int\((A)\) as \( \bigcup\{W | W \in m_X, W \subseteq A\} \) and the \( m_X \)-closure of \( A \) abbreviate \( m_X \)-Cl\((A)\) as \( \bigcap\{F | A \subseteq F, X \setminus F \in m_X\} \). An immediate consequence of the above Definition is the following theorem.

**Theorem 2.1.** Let \((X,m_X)\) be an \( m \)-space and \( A \) a subset of \( X \). Then \( x \in m_X \)-Cl\((A)\) if and only if \( U \cap A \neq \emptyset \) for every \( U \in m_X \) containing \( x \).

And satisfying the following properties:
(i) \( m_X \)-Cl\((m_X \text{-Cl}(A)) = m_X \text{-Cl}(A) \).
(ii) \( m_X \text{-Int}(m_X \text{-Int}(A)) = m_X \text{-Int}(A) \).
(iii) \( m_X \text{-Int}(X \setminus A) = X \setminus m_X \text{-Cl}(A) \).
(iv) \( m_X \text{-Cl}(X \setminus A) = X \setminus m_X \text{-Int}(A) \).
(v) If \( A \subseteq B \) then \( m_X \text{-Cl}(A) \subseteq m_X \text{-Cl}(B) \).
(vi) \( m_X \text{-Cl}(A \cup B) \subseteq m_X \text{-Cl}(A) \cup m_X \text{-Cl}(B) \).
(vii) \( A \subseteq m_X \text{-Cl}(A) \) and \( m_X \text{-Int}(A) \subseteq A \).

**Proof.** Follows from Definition 2.1. \( \square \)

**Definition 2.2.** Let \((X,m_X)\) be an \( m \)-space. We say that \( A \subseteq X \) is an \( m_X \)-semiflame open set if there exists \( U \in m_X \) such that \( U \subseteq A \subseteq m_X \text{-Cl}(U) \). Also we say that \( A \subseteq X \) is \( m_X \)-semiflame closed if \( X \setminus A \) is \( m_X \)-semiflame open.

**Definition 2.3.** Let \((X,m_X)\) be an \( m \)-space. We say that \( A \subseteq X \) is an \( m_X \)-preopen set if \( A \subseteq m_X \text{-Int}(m_X \text{-Cl}(A)) \). Also we say that \( A \subseteq X \) is \( m_X \)-preclosed if \( X \setminus A \) is \( m_X \)-preopen.

We denote by \( SO(X,m_X) \) (resp. \( SC(X,m_X) \), \( PO(X,m_X) \), \( PC(X,m_X) \)) the collection of all \( m_X \)-semiflame open (resp. \( m_X \)-semiflame closed, \( m_X \)-preopen, \( m_X \)-preclosed) sets of \((X,m_X)\).
Observe that when $m_X$ is a topology on $X$, then $m_X$-Cl$(A) = \text{Cl}(A)$ for every subset $A$ of $X$.

**Definition 2.4.** Let $(X, m_X)$ be an $m$-space and $B \subseteq X$.

(i) The $m_X$-semiclosure of $B$ denoted by $m_X$-sCl$(B)$ is defined to be the intersection of all $m_X$-semiclosed sets of $(X, m_X)$ containing $B$.

(ii) The $m_X$-preclosure of $B$ denoted by $m_X$-pCl$(B)$ is defined to be the intersection of all $m_X$-preclosed sets of $(X, m_X)$ containing $B$.

We can observe that the $m_X$-semiclosure of a subset $B$ of $X$ satisfies the following properties:

(i) $m_X$-sCl$(\emptyset) = \emptyset$.

(ii) $m_X$-sCl$(X) = X$.

(iii) If $A \subseteq B$ then $m_X$-sCl$(B) \subseteq m_X$-sCl$(B)$.

(iv) If $\emptyset \neq B \neq X$. Then $m_X$-sCl$(B)$ is not necessarily an $m_X$-semiclosed set.

(v) $m_X$-sCl$(X \setminus A) = X \setminus m_X$-sInt$(A)$.

(vi) $m_X$-sInt$(X \setminus A) = X \setminus m_X$-sCl$(A)$.

In the same way, the $m_X$-preclosure of a subset $B$ of $X$ satisfies the following properties:

(i) $m_X$-pCl$(\emptyset) = \emptyset$.

(ii) $m_X$-pCl$(X) = X$.

(iii) If $A \subseteq B$ then $m_X$-pCl$(B) \subseteq m_X$-pCl$(B)$.

(iv) If $\emptyset \neq B \neq X$. Then $m_X$-pCl$(B)$ is not necessarily an $m_X$-pre closed set.

(v) $m_X$-pCl$(X \setminus A) = X \setminus m_X$-pInt$(A)$.

(vi) $m_X$-pInt$(X \setminus A) = X \setminus m_X$-pCl$(A)$.

At this point there is a natural question. There exist any conditions on the $m$-structure of $X$ in order to guarantee that the $m_X$-sCl$(B)$ is an $m_X$-semiclosed set. At this point we introduce the following property.

**Definition 2.5.** Let $(X, m_X)$ be an $m$-space. We say that $m_X$ to have the property of Maki, if the union of any family of elements of $m_X$ is in $m_X$.

Observe that any collection $\emptyset \neq J \subseteq P(X)$, always is contained in an $m$-structure that have the property of Maki, as we know, $m_X(J) = \{\emptyset, X\} \cup \{\bigcup_{M \in J} M : \emptyset \neq J \subseteq J\}$. In particular, when $J = m_X$, we denote by $m_X = m_X(J)$. Clearly $m_X = m_X$, if $m_X$ have the property of Maki. Note that if $m_X$ is an $m$-structure and $Y \subseteq X$, then $\{M \cap Y : M \in m_X\}$ is an $m$-subspace of $(X, m_X)$.

**Theorem 2.2 (8).** Let $(X, m_X)$ be an $m$-space and $m_X$ satisfying the property of Maki. For a subset $A$ of $X$, the following properties hold:

(i) $A \in m_X$ if and only if $m_X$-Int$(A) = A$.

(ii) $A$ is $m_X$-closed if and only if $m_X$-Cl$(A) = A$. 
(iii) $m_X$-Int$(A) \subseteq m_X$ and $m_X$-Cl$(A)$ is $m_X$-closed.

**Proof.** Follows from the definition of $m_X$-closed, $m_X$-Interior and the property of Maki.

In general the $m_X$-open sets and the $m_X$-semiopen sets are not stable for the union. Nevertheless, for certain $m_X$-structure, the class of $m_X$-semiopen sets are stable under union of sets, like it is demonstrated in the following lemma.

**Lemma 2.1.** Let $m_X$ be an $m$-structure on $X$ which satisfy the property of Maki. If $A_i \in SO(X, m_X)$ for each $i \in I$, then $\bigcup_{i \in I} A_i \in SO(X, m_X)$.

**Proof.** Suppose that $m_X$ has the property of Maki, and $A_i \in SO(X, m_X)$ for each $i \in I$. For each $i \in I$, there exists a set $U_i \in m_X$ such that $U_i \subseteq A_i \subseteq U_i$, in consequence, $\bigcup_{i \in I} U_i \subseteq \bigcup_{i \in I} A_i \subseteq \bigcup_{i \in I} m_X$-Cl$(U_i)$. Since $m_X$-Cl is a monotone operator, then $\bigcup_{i \in I} m_X$-Cl$(U_i) \subseteq m_X$-Cl$(\bigcup_{i \in I} U_i)$; and $\bigcup_{i \in I} U_i \in m_X$, because $m_X$ has the property of Maki. In consequence, $\bigcup_{i \in I} U_i \in m_X$ and $\bigcup_{i \in I} U_i \subseteq \bigcup_{i \in I} A_i \subseteq m_X$-Cl$(\bigcup_{i \in I} U_i)$ and consequently $\bigcup_{i \in I} A_i \in SO(X, m_X)$.

As a consequence of the definition of the $m_X$-semiclosure, we have the following.

**Theorem 2.3.** Let $m_X$ be an $m$-structure on $X$. Then:

(i) $x \in m_X$-sCl$(A)$ if and only if $U \cap A \neq \emptyset$ for every $m_X$-semi open set $U$ containing $x$. In the case that $m_X$ satisfy the property of Maki, then

(ii) $A$ is an $m_X$-semiclosed set if and only if $A = m_X$-sCl$(A)$.

**Theorem 2.4.** Let $(X, m_X)$ be an $m$-space and $A \subseteq X$. If $m_X$ satisfy the property of Maki. Then $m_X$-sCl$(A) = A \cup m_X$-Int$(m_X$-Cl$(A))$.

**Proof.** Since $m_X$ satisfies the property of Maki, then $m_X$-sCl$(A)$ is an $m_X$-semiclosed set, using Definition 2.2, we obtain that $m_X$-Int$(m_X$-Cl$(m_X$-sCl$(A))) \subseteq m_X$-sCl$(A)$. Therefore $m_X$-Int$(m_X$-Cl$(A)) \subseteq m_X$-sCl$(A)$ and follows that $A \cup m_X$-Int$(m_X$-Cl$(A)) \subseteq m_X$-sCl$(A)$.

The opposite inclusion, we observe that $m_X$-Int$(m_X$-Cl$(A \cup m_X$-Int$(m_X$-Cl$(A)))) = m_X$-Int$(m_X$-Cl$(A) \cup m_X$-Cl$(m_X$-Int$(m_X$-Cl$(A)))) \subseteq (m_X$-Cl$(A)) \cup m_X$-Int$(m_X$-Cl$(m_X$-Int$(m_X$-Cl$(A)))) = m_X$-Cl$(A) \cup m_X$-Int$(m_X$-Cl$(A)) = m_X$-Cl$(A)$. Thus $m_X$-Int$(m_X$-Cl$(A \cup m_X$-Int$(m_X$-Cl$(A)))) \subseteq m_X$-Int$(m_X$-Cl$(A)) \subseteq A \cup m_X$-Int$(m_X$-Cl$(A))$. Follows that $m_X$-Int$(m_X$-Cl$(A \cup m_X$-Int$(m_X$-Cl$(A)))) \subseteq A \cup m_X$-Int$(m_X$-Cl$(A))$. In consequence, by Definition 2.2, $A \cup m_X$-Int$(m_X$-Cl$(A))$ is an $m_X$-semiclosed set and so $m_X$-sCl$(A) \subseteq A \cup m_X$-Int$(m_X$-Cl$(A))$.

The following example shows that if the maki condition is removed in the previous theorem the equality is not necessarily true.

**Example 2.1.** Let $X = \mathbb{N}$. Also, define the $m$-structure on $X$ as follows: $m_X = \{\emptyset, \mathbb{N}, P(\{2n : n \in \mathbb{N}\}), \{1\}\}$. Then, the $m_X$-closed sets $\emptyset, \mathbb{N}, P(\{2n :
$n \in \mathbb{N})^c$ and $\mathbb{N} - \{1\}$. Also, $SO(X, m_X) = \{\emptyset, N, P(\{2n : n \in \mathbb{N}\}), \{1\}, F\}$, where $F \cap \{2n : n \in \mathbb{N}\} \neq \emptyset$. If we take $A = \{3\}$, then $m_X-\text{sCl}(A) = \{3\}$, $m_X-\text{Cl}(A) = \{2n + 1 : n \in \mathbb{N}\}$ and $m_X-\text{Int}(\{2n + 1 : n \in \mathbb{N}\}) = \{1\}$. It is clear that $A \cup m_X-\text{Int}(m_X-\text{Cl}(A)) = \{1, 3\} \cup \{3\} = m_X-\text{sCl}(A)$. In consequence, $m_X-\text{sCl}(A) \subset A \cup m_X-\text{Int}(m_X-\text{Cl}(A))$.

**Theorem 2.5.** Let $(X, m_X)$ be an $m$-space and $A \subseteq X$. If $m_X$ satisfies the property of Maki. Then

(i) $m_X-\text{sInt}(A) = A \cap m_X-\text{Cl}(m_X-\text{Int}(A))$.

(ii) $m_X-\text{pCl}(A) = A \cup m_X-\text{Cl}(m_X-\text{Int}(A))$.

(iii) $m_X-\text{pInt}(A) = A \cap m_X-\text{Int}(m_X-\text{Cl}(A))$.

**Proof.** (i) Follows from Theorems 2.3 and 2.4.

(ii) The proof is similar to the proof of Theorem 2.4.

(iii) Follows from (ii). \(\square\)

**Definition 2.6.** Let $(X, m_X)$ be an $m$-space. We say that $A \subseteq X$ is an:

(i) $m_X$-regular open set if $A = m_X-\text{Int}(m_X-\text{Cl}(A))$. Also we say that $A \subseteq X$ is an $m_X$-regular closed set if $X \setminus A$ is an $m_X$-regular open set.

(ii) $m_X$-$\pi$-open set if $A$ is the finite union of $m_X$-regular open sets.

(iii) $m_X$-$\pi$-closed set if the $m_X$-$\text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ belong to $m_X$.

(iv) $m_X$-$\pi\text{g}$-$\pi$-closed set if the $m_X$-$\text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $m_X$-$\pi$-open in $X$.

(v) $m_X$-$\text{gp}$-$\pi$-closed set if the $m_X$-$\text{pCl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $m_X$-$\pi$-open in $X$.

(vi) $m_X$-$\text{gp}$-$\pi$-closed set if the $m_X$-$\text{pCl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $m_X$-$\pi$-open in $X$.

(vii) $m_X$-$\text{gs}$-$\pi$-closed set if the $m_X$-$\text{sCl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ belong to $m_X$.

(viii) $m_X$-$\text{sg}$-$\pi$-closed set if the $m_X$-$\text{sCl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is an $m_X$-semiopen set in $X$.

(ix) $m_X$-$\pi$-$\text{gs}$-$\pi$-closed set if the $m_X$-$\text{sCl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $m_X$-$\pi$-open in $X$.

The class of all $m_X$-regular open (resp. $m_X$-$X$-open) sets of an $m$-space $(X, m_X)$ is denoted by $\text{RO}(X, m_X)$ (resp. $\text{PIO}(X, m_X)$).

**Example 2.2.** Let $X = \{a, b, c, d\}$. Define the $m$-structure on $X$ as follows: $m_X = \{\emptyset, X, \{a\}, \{b\}, \{a, c\}\}$. We obtain that $SO(X, m_X) = \{\emptyset, X, \{a\}, \{b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{a, c, d\}\}$, $PO(X, m_X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, c\}, \{a, b, c\}\}$, $RO(X, m_X) = \{\emptyset, X, \{b\}, \{a, c\}\}$ and $PIO(X, m_X) = \{\emptyset, X, \{b\}, \{a, c\}, \{a, b, c\}\}$.

**Example 2.3.** Let $X = \{a, b, c\}$. Define the $m$-structure on $X$ as follows: $m_X = \{\emptyset, X, \{a\}, \{b\}\}$. Then, we obtain that: $SO(X, m_X) = \{\emptyset, X, \{a\}, \{b\}$,
In general the notions \( \emptyset = \{ \emptyset \} \).

Observation 2.2. The converses in Theorem 2.6 is not necessarily true as Example 2.7. Let \( \emptyset \).

\[ \begin{align*}
\text{Example 2.4. Let } &X = \{a, b, c, d\}. \text{ Define the } m\text{-structure on } X \text{ as follows } \\
m_X = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}. \text{ Then, we obtain that: } \\
SO(X, m_X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}, \\
RO(X, m_X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}\} \\
PO(X, m_X) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}\text{ and } \\
PO(X, m_X) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}. \\
\end{align*} \]

\[ \begin{align*}
\text{Example 2.5. Let } &X = \{a, b, c, d\}. \text{ Define the } m\text{-structure on } X \text{ as follows } \\
m_X = \{\emptyset, X, \{a\}, \{b\}, \{a, b, d\}, \{a, b, c\}\}. \text{ Then, we obtain that: } \\
SO(X, m_X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, d\}, \{a, b, c\}\}, \\
RO(X, m_X) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \\
PIO(X, m_X) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}. \\
\end{align*} \]

\[ \begin{align*}
\text{Example 2.6. Let } &X = \{a, b, c, d\}. \text{ Define the } m\text{-structure on } X \text{ as follows } \\
m_X = \{\emptyset, X, \{a, b\}, \{a, c\}, \{a, c, d\}, \{b, c\}\}. \text{ Then, we obtain that: } \\
SO(X, m_X) = \{\emptyset, X, \{a, c\}, \{a, c, d\}, \{b, c\}\}, \\
PO(X, m_X) = \{\emptyset, X, \{a\}, \{b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c\}\}, \\
RO(X, m_X) = \{\emptyset, X, \{a\}, \{b\}, \{a, c\}, \{a, c, d\}, \{b, c\}\}, \text{ and } \\
PO(X, m_X) = \{\emptyset, X, \{a\}, \{b\}, \{a, c\}, \{a, c, d\}, \{b, c\}\}. \\
\end{align*} \]

\[ \begin{align*}
\text{Example 2.7. Let } &X = \{a, b, c, d\}. \text{ Define the } m\text{-structure on } X \text{ as follows } \\
m_X = \{\emptyset, X, \{a\}, \{b\}, \{c\}\}. \text{ Then, we obtain that: } \\
SO(X, m_X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}\}, \\
PO(X, m_X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}\}, \\
RO(X, m_X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}\}, \text{ and } \\
PO(X, m_X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}\}. \\
\end{align*} \]

Observation 2.1. In general the notions \( m_X\)-gs-closedness and \( m_X\)-sg-closedness are independent as we can see:

(i) In Example 2.2, the set \( \{a, c\} \) is \( m_X\)-gs-closed but not \( m_X\)-sg-closed.

(ii) In Example 2.4, the set \( \{a, b\} \) is \( m_X\)-sg-closed but not \( m_X\)-gs-closed.

Theorem 2.6. The following properties are true for a subset \( A \) of an \( m\)-space \((X, m_X)\).

(i) If \( A \) is \( m_X\)-closed, then it is \( m_X\)-semiclosed.

(ii) If \( A \) is \( m_X\)-closed, then it is \( m_X\)-preclosed.

(iii) If \( A \) is \( m_X\)-semiclosed, then it is \( m_X\)-gs-closed.

(iv) If \( A \) is \( m_X\)-g-closed, then it is \( m_X\)-gs-closed.

(v) If \( A \) is \( m_X\)-g-closed, then it is \( m_X\)-gp-closed.

(vi) If \( A \) is \( m_X\)-g-closed, then it is \( m_X\)-ggs-closed.

(vii) If \( A \) is \( m_X\)-ggs-closed, then it is \( m_X\)-ggp-closed.

Observation 2.2. The converses in Theorem 2.6 is not necessarily true as we can see.

(i) In Example 2.2, the set \( \{b\} \) is \( m_X\)-semiclosed but not \( m_X\)-closed.
(ii) In Example 2.6, the set \( \{b\} \) is \( m_X \)-preclosed but not \( m_X \)-closed.

(iii) In Example 2.2, the set \( \{c\} \) is \( m_X \)-gs-closed but not \( m_X \)-semiclosed.

(iv) In Example 2.2, the set \( \{b\} \) is \( m_X \)-gs-closed but not \( m_X \)-g-closed.

(v) In Example 2.2, the set \( \{a, d\} \) is \( m_X \)-gp-closed but not \( m_X \)-g-closed.

(vi) In Example 2.2, the set \( \{a\} \) is \( m_X \)-\( \pi \)gs-closed but not \( m_X \)-\( \pi \)g-closed.

(vii) In Example 2.7, the set \( \{a\} \) is \( m_X \)-gp-closed but not \( m_X \)-\( \pi \)g-closed.

The following Theorem improve the Remark 2.1 in [1].

**Theorem 2.7.** Let \((X, m_X)\) be an \( m \)-space where \( m_X \) satisfy the property of Maki and \( A \subseteq X \). The following properties are hold:

(i) If \( A \) is \( m_X \)-gs-closed, then \( A \) is \( m_X \)-\( \pi \)gs-closed.

(ii) If \( A \) is \( m_X \)-gp-closed, then \( A \) is \( m_X \)-\( \pi \)gp-closed.

(iii) If \( A \) is \( m_X \)-g-closed, then \( A \) is \( m_X \)-\( \pi \)g-closed.

(iv) If \( A \) is \( m_X \)-sg-closed, then \( A \) is \( m_X \)-\( \pi \)-sg-closed.

**Observation 2.3.** If the condition of Maki is dropped in Theorem 2.7, the result is not necessarily true as we can see as follows:

(i) Take the set \( \{a, b\} \) in Example 2.3.

(ii) Take the set \( \{a, b, c\} \) in Example 2.2.

(iii) Take the set \( \{a, b\} \) in Example 2.3.

(iv) Take the set \( \{a, b\} \) in Example 2.3.

**Observation 2.4.** In Theorem 2.7, none of the implications are reversible.

(i) In Example 2.5, the set \( \{a, b, c\} \) is \( m_X \)-\( \pi \)gs closed but not \( m_X \)-gs-closed.

(ii) In Example 2.6, the set \( \{a, b, c\} \) is \( m_X \)-\( \pi \)gp-closed but not \( m_X \)-gp closed.

(iii) In Example 2.5, the set \( \{a, c\} \) is \( m_X \)-\( \pi \)g-closed but not \( m_X \)-g-closed.

(iv) In Example 2.6, the set \( \{c\} \) is \( m_X \)-\( \pi \)-sg-closed but not \( m_X \)-sg-closed.

**Observation 2.5.** The notion of \( m_X \)-gs-closedness \( m_X \)-gp-closedness are independent.

(i) In Example 2.5, the set \( \{a\} \) is \( m_X \)-gs-closed but not \( m_X \)-gp-closed.

(ii) In Example 2.6, the set \( \{c\} \) is \( m_X \)-gp-closed but not \( m_X \)-gs-closed.

**Observation 2.6.** The notions \( m_X \)-\( \pi \)gs-closed is different from the notion of \( m_X \)-\( \pi \)gp-closed set as we can see:

(i) In Example 2.5, the set \( \{b\} \) is \( m_X \)-\( \pi \)gs-closed but not \( m_X \)-\( \pi \)gp-closed.

(ii) In Example 2.7, the set \( \{a\} \) is \( m_X \)-\( \pi \)gp closed but not \( m_X \)-\( \pi \)gs-closed.

**Definition 2.7.** An \( m \)-space \((X, m_X)\) is said to be:

(i) \( m_X \)-\( T_{1/2} \) space if every \( m_X \)-g-closed set is \( m_X \)-closed in it.

(ii) \( m_X \)-\( sT_{1/2} \) space if every \( m_X \)-gs-closed set is \( m_X \)-semiclosed in it.

(iii) \( m_X \)-\( \pi \)gp\( T_{1/2} \) if every \( m_X \)-\( \pi \)gp-closed set is \( m_X \)-preclosed in it.

The following Theorem characterize the notions \( m_X \)-sg-closed, \( m_X \)-g-closed, \( m_X \)-gs-closed, \( m_X \)-gp-closed, \( m_X \)-\( \pi \)g-closed, \( m_X \)-\( \pi \)gs-closed and \( m_X \)-\( \pi \)gp-closed in \( m \)-any structure on \( X \).
Theorem 2.8. The following results are true for a subset $A$ of an $m$-space $(X, m_X)$:

(i) $A$ is $m_X$-sg-closed if and only if $m_X$-$Cl(A) \subseteq m_X$-$sKer(A)$.

(ii) $A$ is $m_X$-g-closed if and only if $m_X$-$Cl(A) \subseteq m_X$-$sKer(A)$.

(iii) $A$ is $m_X$-gs-closed if and only if $m_X$-$sCl(A) \subseteq m_X$-$sKer(A)$.

(iv) $A$ is $m_X$-gp-closed if and only if $m_X$-$sCl(A) \subseteq m_X$-$sKer(A)$.

(v) $A$ is $m_X$-$\pi$g-closed if and only if $m_X$-$sCl(A) \subseteq m_X$-$g$-$sKer(A)$.

(vi) $A$ is $m_X$-$\pi$gs-closed if and only if $m_X$-$sCl(A) \subseteq m_X$-$g$-$sKer(A)$.

(vii) $A$ is $m_X$-$\pi$gp-closed if and only if $m_X$-$sCl(A) \subseteq m_X$-$g$-$sKer(A)$.

Where $m_X$-$Ker(A)$ (resp. $m_X$-$sKer(A)$, $m_X$-$pKer(A)$, $m_X$-$\pi$-$Ker(A)$) is defined as the intersection of all $m_X$-open sets containing $A$.

Proof. We shall prove the result (i) only, because the proof of all of them are similar. We are going to prove that: $A \subseteq X$ is an $m_X$-sg-closed set if and only if $m_X$-$sCl(A) \subseteq m_X$-$sKer(A)$.

Let $D = \{S : S \subseteq X, A \subseteq S, S \in SO(X, m_X)\}$. Then $m_X$-$sKer(A) = \bigcap_{S \in D} S$. Observe that $S \in D$ implies that $A \subseteq S$ follows $m_X$-$sCl(A) \subseteq S$ for all $S \in D$.

In consequence, $m_X$-$sCl(A) \subseteq m_X$-$sKer(A)$. If $m_X$-$sCl(A) \subseteq m_X$-$sKer(A)$, take $S \in SO(X, m_X)$ such that $A \subseteq S$, then by hypothesis $m_X$-$sCl(A) \subseteq m_X$-$sKer(A) \subseteq S$. This shows that $A$ is $m_X$-sg-closed. \qed

It is important to know that the proof of the results of the items of the above Theorem 2.7, only the set $D$ can be changed in an addequate form as follows: In (ii), (iii) and (iv) take $D = \{S : S \subseteq X, A \subseteq S : S \in m_X\}$. In (v), (vi) and (vii) take $D = \{S : S \subseteq X, A \subseteq S : S$ is an $m_X$-$\pi$-open$\}$.

The following Theorem 2.8, characterize the $m_X$-gs-closed sets, $m_X$-sg-closed sets, $m_X$-gp-closed sets, $m_X$-gs-closed sets, $m_X$-$\pi$gp-closed sets, $m_X$-$\pi$gp-closed sets, where the $m$-structure on $X$ satisfies the property of Maki. In this form, we generalize the Theorems 2.3 and 2.4 given in [1].

Theorem 2.9. Let $m_X$ be an $m$-structure on $X$ satisfying the property of Maki and $A \subseteq X$. Then:

(i) $A$ is $m_X$-gs-closed if and only if there does not exists a nonempty $m_X$-closed set $F F \subseteq m_X$-$sCl(A) \setminus A$.

(ii) $A$ is an $m_X$-gs-closed set $F \subseteq m_X$-$sCl(A) \setminus A$.

(iii) $A$ is an $m_X$-g-closed set $F \subseteq m_X$-$g$-$sCl(A) \setminus A$.

(iv) $A$ is an $m_X$-$\pi$g-closed set $F \subseteq m_X$-$\pi$-$sCl(A) \setminus A$.

(v) $A$ is an $m_X$-$\pi$gp-closed set $F \subseteq m_X$-$\pi$-$pCl(A) \setminus A$.

(vi) $A$ is an $m_X$-$\pi$gp-closed set $F \subseteq m_X$-$\pi$-$pCl(A) \setminus A$. 

(vii) \(A\) is an \(m_X\)-\(\pi\)-\(gs\)-closed if and only if there does not exists a nonempty \(m_X\)-\(\pi\)-closed set \(F\) \(F \subseteq m_X\)-\(s\)\(Cl(A) \setminus A\).

**Proof.** (i) Suppose that \(A\) is an \(m_X\)-\(gs\)-closed and let \(F \subseteq X\) be an \(m_X\)-closed set such that \(F \subseteq m_X\)-\(s\)\(Cl(A) \setminus A\). It follows that, \(A \subseteq X \setminus F\) and \(X \setminus F\) is an \(m_X\)-open set, since \(A\) is an \(m_X\)-\(gs\)-closed, we have that \(m_X\)-\(s\)\(Cl(A) \subseteq X \setminus F\) and \(F \subseteq X \setminus m_X\)-\(s\)\(Cl(A)\). It follows that,

\[
F \subseteq m_X\)-\(s\)\(Cl(A) \cap (X \setminus m_X\)-\(s\)\(Cl(A)) = \emptyset,
\]

implying that \(F = \emptyset\). Reciprocally, if \(A \subseteq U\) and \(U\) is an \(m_X\)-open set, then \(m_X\)-\(s\)\(Cl(A) \cap (X \setminus U) \subseteq m_X\)-\(s\)\(Cl(A) \cap (X \setminus A) = m_X\)-\(s\)\(Cl(A) \setminus A\). Since \(m_X\)-\(s\)\(Cl(A) \setminus A\) does not contain any nonempty \(m_X\)-closed, we obtain that \(m_X\)-\(s\)\(Cl(A) \cap (X \setminus U) = \emptyset\). It follows that \(m_X\)-\(s\)\(Cl(A) \subseteq U\) and hence \(A\) is \(m_X\)-\(gs\)-closed in \((X, m_X)\).

(ii) Suppose that \(A\) is an \(m_X\)-\(sg\)-closed and let \(F\) be an \(m_X\)-semiopen set of \((X, m_X)\) such that \(F \subseteq m_X\)-\(s\)\(Cl(A) \setminus A\). It follows that, \(A \subseteq X \setminus F\) and \(X \setminus F\) is an \(m_X\)-semiopen set, since \(A\) is an \(m_X\)-\(sg\)-closed, we have that \(m_X\)-\(s\)\(Cl(A) \subseteq X \setminus F\) and \(F \subseteq X \setminus m_X\)-\(s\)\(Cl(A)\). Follows that,

\[
F \subseteq m_X\)-\(s\)\(Cl(A) \cap (X \setminus m_X\)-\(s\)\(Cl(A)) = \emptyset,
\]

implying that \(F = \emptyset\). Reciprocally, if \(A \subseteq U\) and \(U\) is an \(m_X\)-semiopen set, then \(m_X\)-\(s\)\(Cl(A) \cap (X \setminus U) \subseteq m_X\)-\(s\)\(Cl(A) \cap (X \setminus A) = m_X\)-\(s\)\(Cl(A) \setminus A\). Since \(m_X\)-\(s\)\(Cl(A) \setminus A\) does not contain any nonempty \(m_X\)-semiopen sets, we obtain that \(m_X\)-\(s\)\(Cl(A) \cap (X \setminus U) = \emptyset\). It follows that \(m_X\)-\(s\)\(Cl(A) \subseteq U\) and hence \(A\) is an \(m_X\)-\(gs\)-closed.

(iv) Suppose that \(A\) is an \(m_X\)-\(\pi\)g-closed and let \(F \subseteq X\) be an \(m_X\)-\(\pi\)-closed set such that \(F \subseteq m_X\)-\(Cl(A) \setminus A\). It follows that, \(A \subseteq X \setminus F\) and \(X \setminus F\) is an \(m_X\)-\(\pi\)-open set, since \(A\) is an \(m_X\)-\(\pi\)g-closed, we have that \(m_X\)-\(Cl(A) \subseteq X \setminus F\) and \(F \subseteq X \setminus m_X\)-\(Cl(A)\). Follows that,

\[
F \subseteq m_X\)-\(Cl(A) \cap (X \setminus m_X\)-\(Cl(A)) = \emptyset,
\]

implying that \(F = \emptyset\). Reciprocally, if \(A \subseteq U\) and \(U\) is an \(m_X\)-\(\pi\)-openset, then \(m_X\)-\(Cl(A) \cap (X \setminus U) \subseteq m_X\)-\(Cl(A) \cap (X \setminus A) = m_X\)-\(Cl(A) \setminus A\). Since \(m_X\)-\(Cl(A) \setminus A\) does not contain any nonempty \(m_X\)-\(\pi\)-closed sets, we obtain that \(m_X\)-\(Cl(A) \cap (X \setminus U) = \emptyset\), and this implies that \(m_X\)-\(Cl(A) \subseteq U\) in consequence \(A\) is an \(m_X\)-\(\pi\)g-closed.

The proof of (iii), (v), (vi) and (vii) are similar. \(\Box\)

We can observe that if in Theorem 2.9 the condition of Maki is omitted then the result can be false, as we can see in the following example.

**Example 2.8.** (i) In Example 2.4, the set \(\{a, b\}\) is \(sg\)-closed, and \(\{d\}\) is an \(m_X\)-semiopen such that \(\{d\} \subseteq (m_X\)-\(s\)\(Cl(\{a, b\}) \setminus \{a, b\}\). (ii) In Example 2.4, the set \(\{a\}\) is not \(m_X\)-\(g\) closed and there not exists \(m_X\)-closed set \(F\) such that \(F \neq \emptyset\) and \(F \subseteq m_X\)-\(Cl(A) \setminus A\).
(iii) In Example 2.2, the set \( \{a\} \) is not \( m_X \)-gs-closed, and there does not exists a \( m_X \)-closed set \( F \) such that \( F \neq \emptyset \) and \( F \subseteq m_X\text{-sCl}(\{a\}) \setminus \{a\} \).

(iv) In Example 2.7 the set \( \{c\} \) is not \( m_X \)-gp-closed, and there does not exists a \( m_X \)-closed set \( F \) such that \( F \neq \emptyset \) and \( F \subseteq m_X\text{-pCl}(\{c\}) \setminus \{c\} \).

**Theorem 2.10.** Let \((X,m_X)\) be an \( m \)-space satisfying the property of Maki and \( A \subseteq X \). Then the following properties are equivalent:

(i) \( A \) is \( m_X \)-\( \pi \)-open and \( m_X \)-\( \pi \)-gs-closed.

(ii) \( A \) is \( m_X \)-regular open.

**Proof.** (i) \( \Rightarrow \) (ii): Since \( A \) is \( m_X \)-\( \pi \)-gs-closed then \( m_X \text{-sCl}(A) \subseteq A \) because \( A \) is \( m_X \)-open. Using Theorem 2.4, \( m_X \text{-sCl}(A) = A \cup m_X\text{-Int}(m_X\text{-Cl}(A)) \). We obtain that \( m_X\text{-Int}(m_X\text{-Cl}(A)) \subseteq A \). Using the hypothesis on \( m_X \), we obtain that, \( A \) is \( m_X \)-open, in consequence \( A \) is \( m_X \)-preopen. Follows that \( A \subseteq m_X\text{-Int}(m_X\text{-Cl}(A)) \) and therefore \( m_X\text{-Int}(m_X\text{-Cl}(A)) \subseteq A \subseteq m_X\text{-Int}(m_X\text{-Cl}(A)) \). This implies that \( A \) is \( m_X \)-regular open.

(ii) \( \Rightarrow \) (i): Every \( m_X \)-regular open set is \( m_X \)-\( \pi \)-open and \( m_X \)-open therefore \( m_X \)-\( \pi \)-open and \( m_X \)-semiopen. Follows from Theorem 2.4, \( m_X \text{-sCl}(A) = A \). In consequence, \( A \) is \( m_X \)-\( \pi \)-gs-closed. \( \square \)

**Theorem 2.11.** Let \((X,m_X)\) be an \( m \)-space satisfying the property of Maki and \( A \subseteq X \). If \( A \) is \( m_X \)-\( \pi \)-open and \( m_X \)-\( \pi \)-gs-closed then \( A \) is \( m_X \)-semiclosed and hence \( m_X \)-gs-closed.

**Proof.** By Theorem 2.9, \( A \) is \( m_X \)-regular open. Using Theorem 2.4, \( m_X \text{-sCl}(A) = A \), follows that \( A \) is \( m_X \)-semiclosed in consequence \( A \) is \( m_X \)-gs-closed. \( \square \)

It is easy to see in Example 2.2, the set \( \{a,b,c\} \) is \( m_X \)-\( \pi \)-open and \( m_X \)-gs-closed but not \( m_X \)-\( \pi \)-gs-closed.

**Definition 2.8.** A subset \( A \) of an \( m \)-space \((X,m_X)\) is called \( m_X \)-clopenn if \( m_X\text{-Int}(m_X\text{-Cl}(A)) = m_X\text{-Int}(m_X\text{-Cl}(A)) \).

**Theorem 2.12.** Let \((X,m_X)\) be an \( m \)-space satisfying the property of Maki and \( A \subseteq X \). Then the following properties are equivalent:

(i) \( A \) is \( m_X \)-\( \pi \)-clopenn, that is \( m_X \)-\( \pi \)-open and \( m_X \)-\( \pi \)-closed.

(ii) \( A \) is \( m_X \)-\( \pi \)-open, \( m_X \)-clopenn and \( m_X \)-\( \pi \)-gs-closed.

**Proof.** (i) \( \Rightarrow \) (ii): If \( A \) is \( m_X \)-\( \pi \)-clopenn, then \( A \) is \( m_X \)-\( \pi \)-open and \( m_X \)-\( \pi \)-closed, follows that \( A \) is \( m_X \)-open and \( m_X \)-closed and \( m_X \text{-sCl}(A) = A \). In consequence, we obtain that \( A \) is \( m_X \)-\( \pi \)-open, \( m_X \)-clopenn and \( m_X \)-\( \pi \)-gs-closed.

(ii) \( \Rightarrow \) (i): Using the fact that \( A \) is \( m_X \)-\( \pi \)-open and \( m_X \)-\( \pi \)-gs-closed implies by Theorem 2.8, \( A \) is \( m_X \)-regular open. Now if \( A \) is \( m_X \)-clopenn then \( m_X\text{-Int}(m_X\text{-Cl}(A)) = m_X\text{-Int}(m_X\text{-Cl}(A)) = A \). Follows that \( A \) is \( m_X \)-\( \pi \)-clopenn. \( \square \)

**Theorem 2.13.** Let \((X,m_X)\) be an \( m \)-space satisfying the property of Maki and \( A,B \) be subsets of \( X \). Then the following properties hold:
(i) If \( A \) is \( m_X \)-\( g \)-closed and \( A \subset B \subset m_X \-\text{Cl}(A) \), then \( B \) is \( m_X \)-\( g \)-closed.
(ii) If \( A \) is \( m_X \)-\( \pi \)-\( g \)-closed and \( A \subset B \subset m_X \-\text{Cl}(A) \), then \( B \) is \( m_X \)-\( \pi \)-\( g \)-closed.
(iii) If \( A \) is \( m_X \)-\( g \)-\( p \)-closed and \( A \subset B \subset m_X \-p\text{Cl}(A) \), then \( B \) is \( m_X \)-\( g \)-\( p \)-closed.
(iv) If \( A \) is \( m_X \)-\( \pi \)-\( g \)-\( p \)-closed and \( A \subset B \subset m_X \-p\text{Cl}(A) \), then \( B \) is \( m_X \)-\( \pi \)-\( g \)-\( p \)-closed.
(v) If \( A \) is \( m_X \)-\( g \)-\( s \)-closed and \( A \subset B \subset m_X \-s\text{Cl}(A) \), then \( B \) is \( m_X \)-\( g \)-\( s \)-closed.
(vi) If \( A \) is \( m_X \)-\( \pi \)-\( g \)-\( s \)-closed and \( A \subset B \subset m_X \-s\text{Cl}(A) \), then \( B \) is \( m_X \)-\( \pi \)-\( g \)-\( s \)-closed.

\begin{proof}
(i): Let \( A \) be an \( m_X \)-\( g \)-closed subset, \( B \subset U \) where \( U \) is \( m_X \)-open. Since \( A \subset B \), then \( m_X \-\text{Cl}(A) \subset m_X \-\text{Cl}(B) \). It follows that \( m_X \-\text{Cl}(A) \subset U \) and \( B \subset m_X \-\text{Cl}(A) \) implies that \( m_X \-\text{Cl}(B) \subset U \). In consequence \( B \) is \( m_X \)-\( g \)-closed.

The other proofs are similar. \( \square \)
\end{proof}

The proof of the following Theorem 2.14 is easy and hence omitted.

**Theorem 2.14.** Let \((X, m_X)\) be an \( m \)-space satisfying the property of Maki and \( A \subset X \). Then the following properties hold:

(i) \( A \) is \( m_X \)-\( g \)-open if and only if \( F \subset m_X \-\text{Int}(A) \) whenever \( F \) is \( m_X \)-closed and \( F \subset A \).
(ii) \( A \) is \( m_X \)-\( \pi \)-\( g \)-open if and only if \( F \subset m_X \-\text{Int}(A) \) whenever \( F \) is \( m_X \)-\( \pi \)-closed and \( F \subset A \).
(iii) \( A \) is \( m_X \)-\( g \)-\( p \)-open if and only if \( F \subset m_X \-p\text{Int}(A) \) whenever \( F \) is \( m_X \)-closed and \( F \subset A \).
(iv) \( A \) is \( m_X \)-\( \pi \)-\( g \)-\( p \)-open if and only if \( F \subset m_X \-p\text{Int}(A) \) whenever \( F \) is \( m_X \)-\( \pi \)-closed and \( F \subset A \).
(v) \( A \) is \( m_X \)-\( g \)-\( s \)-open if and only if \( F \subset m_X \-s\text{Int}(A) \) whenever \( F \) is \( m_X \)-closed and \( F \subset A \).
(vi) \( A \) is \( m_X \)-\( \pi \)-\( g \)-\( s \)-open if and only if \( F \subset m_X \-s\text{Int}(A) \) whenever \( F \) is \( m_X \)-\( \pi \)-closed and \( F \subset A \).

3. \((m_X, m_Y)\)-Continuous maps and \((m_X, m_Y)\)-Irresolute maps

In this section, we define different forms of continuity and irresoluteness on \( m \)-structures where the notions of \( g \)-closed set, \( gs \)-closed set, \( sg \)-closed set, \( \pi \)-closed set, \( \pi g \)-closed set, \( \pi gs \)-closed set, \( gp \)-closed set, \( \pi gp \)-closed set are involucrate.

**Definition 3.1.** A map \( f: (X, m_X) \to (Y, m_Y) \) is called:

(i) \( \pi \)-\( gs \)-(\( m_X, m_Y \))-continuous if \( f^{-1}(O) \) is \( m_X \)-\( \pi \)-\( gs \)-closed in \((X, m_X)\) for every \( m_Y \)-closed set \( O \) of \((Y, m_Y)\).
(ii) \( \pi \)-(\( m_X, m_Y \))-continuous if \( f^{-1}(O) \) is \( m_X \)-\( \pi \)-closed in \((X, m_X)\) for every \( m_Y \)-closed set \( O \) of \((Y, m_Y)\).

1. \( \pi g \) - \((m_X, m_Y)\) continuous if \( f^{-1}(O) \) is \( m_X \)-\( \pi g \)-closed in \((X, m_X)\) for every \( m_Y \)-closed set \( O \) of \((Y, m_Y)\).
2. $\pi gp(m_X, m_Y)$ continuous if, $f^{-1}(O)$ is $m_X-\pi gp$-closed in $(X, m_X)$ for every $m_Y$-closed set $O$ of $(Y, m_Y)$.

3. $s - (m_X, m_Y)$ continuous if, $f^{-1}(O)$ is $m_X$-semiclosed in $(X, m_X)$ for every $m_Y$-closed set $O$ of $(Y, m_Y)$.

4. $g - (m_X, m_Y)$ continuous if, $f^{-1}(O)$ is $m_X$-g-closed in $(X, m_X)$ for every $m_Y$-closed set $O$ of $(Y, m_Y)$.

5. $gs - (m_X, m_Y)$ continuous if, $f^{-1}(O)$ is $m_X$-gs-closed in $X$ for every $m_Y$-closed set $O$ of $(Y, m_Y)$.

6. $gp - (m_X, m_Y)$ continuous if, $f^{-1}(O)$ is $m_X$-gp-closed in $(X, m_X)$ for every $m_Y$-closed set $O$ of $(Y, m_Y)$.

**Example 3.1.** In the Example 2.3, take $X = Y = \{a, b, c\}, m_X = m_Y = \{\emptyset, X, \{a\}, \{b\}\}$ and $f: (X, m_X) \mapsto (Y, m_Y)$ defined as: $f(a) = f(c) = c$ and $f(b) = a$. Then the function $f$ satisfies all different notions of continuity described in Definition 3.1.

From the above definition, easily we have the following theorem

**Theorem 3.1.** Let $f: (X, m_X) \mapsto (Y, m_Y)$, then:

(i) If $f$ is $(m_X, m_Y)$ continuous, then it is $g-(m_X, m_Y)$-continuous.

(ii) If $f$ is $(m_X, m_Y)$-continuous, then it is $s-(m_X, m_Y)$-continuous.

(iii) If $f$ is $(m_X, m_Y)$-continuous, then it is $gp-(m_X, m_Y)$-continuous.

(iv) If $f$ is $g-(m_X, m_Y)$-continuous, then it is $gp-(m_X, m_Y)$-continuous.

(v) If $f$ is $s-(m_X, m_Y)$-continuous, then it is $gs-(m_X, m_Y)$-continuous.

(vi) If $f$ is $g-(m_X, m_Y)$-continuous, then it is $gs-(m_X, m_Y)$-continuous.

(vii) If $f$ is $g-(m_X, m_Y)$-continuous, then it is $gp-(m_X, m_Y)$-continuous.

and none of them are reversible.

**Proof.** The proof follows from Theorem 2.6. □

**Theorem 3.2.** Let $f: (X, m_X) \mapsto (Y, m_Y)$, where $m_X$ satisfy the property of Maki, then If $f$ is $\pi -(m_X, m_Y)$-continuous then $f$ is $(m_X, m_Y)$-continuous.

**Proof.** The proof follows from the fact that any $\pi$-closed set is closed in any $m$-structure. □

**Example 3.2.** In the Example 2.3, take $X = Y = \{a, b, c\}, m_X = m_Y = \{\emptyset, X, \{a\}, \{b\}\}$ and $f: (X, m_X) \mapsto (Y, m_Y)$ defined as: $f(a) = f(b) = a$ and $f(c) = c$. Then $f$ is $\pi -(m_X, m_Y)$-continuous but does not is $(m_X, m_Y)$-continuous.

In the case that the $f: (X, m_X) \mapsto (Y, m_Y)$ is a map, where $m_X$ satisfy the condition of Maki, we have the following Theorem.

**Theorem 3.3.** Let $f: (X, m_X) \mapsto (Y, m_Y)$, where $m_X$ satisfies the condition of Maki then:

1. If $f$ is $g-(m_X, m_Y)$-continuous, then $f$ if $\pi g-(m_X, m_Y)$-continuous.

2. If $f$ is $gs-(m_X, m_Y)$-continuous, then $f$ if $\pi gs-(m_X, m_Y)$-continuous.
3. If \( f \) is \( gp-(m_X,m_Y) \)-continuous, then \( f \) if \( \pi-gp-(m_X,m_Y) \)-continuous.

and none of them are reversible.

**Proof.** The proof follows from Theorem 2.7.

In the following example, we shows that if the condition of Maki on \( m_X \) is omitted, then the Theorem 3.3 can be false

**Example 3.3.** In the Example 2.3, take \( X = Y = \{a,b,c\}, m_X = m_Y = \{\emptyset,X,\{a\},\{b\}\} \) and \( f : (X,m_X) \mapsto (Y,m_Y) \) defined as: \( f(a) = c, f(b) = a \) and \( f(c) = b \). Then:

(i) \( f \) is \( g-(m_X,m_Y) \)-continuous but not \( \pi-g-(m_X,m_Y) \)-continuous.

(ii) \( f \) is \( gs-(m_X,m_Y) \)-continuous but not \( \pi gs-(m_X,m_Y) \)-continuous.

**Example 3.4.** In the Example 2.3, take \( X = \{a,b,c\}, m_X = \{\emptyset,X,\{a\},\{b\}\} \) and \( Y = \{x,y\}, and m_Y = \{\emptyset,Y,\{x\}\} \), \( f : (X,m_X) \mapsto (Y,m_Y) \) defined as: \( f(a) = f(c) = x \) and \( f(b) = y \). Then \( f \) is \( \pi-gp-(m_X,m_Y) \)-continuous but not \( \pi-gs-(m_X,m_Y) \)-continuous and \( \pi-gp-(m_X,m_Y) \)-continuous.

**Example 3.5.** In the Example 2.7, take \( X = \{a,b,c,d\}, m_X = \{\emptyset,X,\{a\},\{b\},\{a,c\},\{b\},\{c\}\} \) and \( Y = \{x,y\}, and m_Y = \{\emptyset,Y,\{x\}\} \), \( f : (X,m_X) \mapsto (Y,m_Y) \) defined as: \( f(b) = f(c) = f(d) = x \) and \( f(a) = y \). Then \( f \) is \( \pi-gp-(m_X,m_Y) \)-continuous but not \( \pi-gs-(m_X,m_Y) \)-continuous and \( \pi-g-(m_X,m_Y) \)-continuous.

**Example 3.6.** In the Example 2.2, take \( X = \{a,b,c,d\}, m_X = \{\emptyset,X,\{a\},\{b\},\{a,c\}\} \) and \( Y = \{x,y\}, and m_Y = \{\emptyset,Y,\{x\}\} \).

Define \( f : (X,m_X) \mapsto (Y,m_Y) \) as: \( f(a) = f(b) = f(c) = y \) and \( f(d) = y \). Then \( f \) is \( gp-(m_X,m_Y) \)-continuous but none of \( \pi-gp-(m_X,m_Y) \)-continuous, \( \pi-gs-(m_X,m_Y) \)-continuous, \( \pi-g-(m_X,m_Y) \)-continuous and \( \pi-(m_X,m_Y) \)-continuous.

**Definition 3.2.** A map \( f : (X,m_X) \mapsto (Y,m_Y) \) is called:

(i) \( (m_X,m_Y) \)-irresolute if \( f^{-1}(O) \) is \( m_X \)-semiclosed in \( X \) for every \( m_Y \)-semiclosed set \( O \) of \( (Y,m_Y) \).

(ii) \( \pi-(m_X,m_Y) \)-irresolute if, \( f^{-1}(O) \) is \( m_X \)-\( \pi \)-closed in \( (X,m_X) \) for every \( m_Y \)-\( \pi \)-closed set \( O \) of \( (Y,m_Y) \).

(iii) \( \pi-gp-(m_X,m_Y) \)-irresolute if \( f^{-1}(O) \) is \( m_X \)-\( \pi \)-gp-closed in \( (X,m_X) \) for every \( m_Y \)-\( \pi \)-gp-closed set \( O \) of \( (Y,m_Y) \).

(iv) \( \pi-gs-(m_X,m_Y) \)-irresolute if \( f^{-1}(O) \) is \( m_X \)-\( \pi \)-gs-closed in \( (X,m_X) \) for every \( m_Y \)-\( \pi \)-gs-closed set \( O \) of \( (Y,m_Y) \).

(v) \( gs-(m_X,m_Y) \)-irresolute if, \( f^{-1}(O) \) is \( m_X \)-gs-closed in \( (X,m_X) \) for every \( m_Y \)-gs-closed set \( O \) of \( (Y,m_Y) \).

(vi) \( gp-(m_X,m_Y) \)-irresolute if, \( f^{-1}(O) \) is \( m_X \)-gp-closed in \( (X,m_X) \) for every \( m_Y \)-gp-closed set \( O \) of \( (Y,m_Y) \).

(vii) \( g-(m_X,m_Y) \)-irresolute if, \( f^{-1}(O) \) is \( m_X \)-g-closed in \( (X,m_X) \) for every \( m_Y \)-g-closed set \( O \) of \( (Y,m_Y) \).
Example 3.7. In the Example 2.7, take $X = \{a, b, c, d\}, m_X = \{\emptyset, X, \{a, b\}, \{a, c\}, \{a\}, \{b\}, \{c\}\}$. $Y = \{x, y\}$, and $m_Y = \{\emptyset, Y, \{x\}\}$, $f : (X, m_X) \mapsto (Y, m_Y)$ defined as: $f(b) = f(c) = f(d) = x$ and $f(a) = y$. Then $f$ is $\pi$-gp-$(m_X, m_Y)$-irresolute but none of $\pi$-gs-$(m_X, m_Y)$-irresolute and $\pi$-g-$(m_X, m_Y)$-irresolute.

Example 3.8. In the Example 2.4, take $X = \{a, b, c, d\}, m_X = \{\emptyset, X, \{a, b, d\}, \{a, b, c\}, \{a\}, \{b\}\}$. $Y = \{x, y\}$, and $m_Y = \{\emptyset, Y, \{x\}\}$, $f : (X, m_X) \mapsto (Y, m_Y)$ defined as: $f(b) = f(c) = f(d) = x$ and $f(a) = y$. Then $f$ is $\pi$-gs-$(m_X, m_Y)$-irresolute but none of $\pi$-gp-$(m_X, m_Y)$-irresolute, $\pi$-g-$(m_X, m_Y)$-irresolute, $g$-$(m_X, m_Y)$-irresolute and $gp$-$(m_X, m_Y)$-irresolute.

Example 3.9. In the Example 2.2, take $X = \{a, b, c, d\}, m_X = \{\emptyset, X, \{a\}, \{b\}, \{a, c\}\}$. $Y = \{x, y\}$, and $m_Y = \{\emptyset, Y, \{x\}\}$, $f : (X, m_X) \mapsto (Y, m_Y)$ defined as: $f(a) = f(b) = f(c) = y$ and $f(d) = x$. Then $f$ is gp-$(m_X, m_Y)$-irresolute but none of $\pi$-gp-$(m_X, m_Y)$-irresolute, $\pi$-gs-$(m_X, m_Y)$-irresolute, $\pi$-g-$(m_X, m_Y)$-irresolute and $\pi$--$(m_X, m_Y)$-irresolute.

Definition 3.3. A map $f : (X, m_X) \mapsto (Y, m_Y)$ is called:

(i) $(m_X, m_Y)$-pre semiclosed if $f(O)$ is $m_Y$-semiclosed in $(Y, m_Y)$ for all $m_X$-semiclosed set $O$ of $(X, m_X)$.

(ii) $(m_X, m_Y)$-pre semiopen if $f(O)$ is $m_Y$-semiopen in $(Y, m_Y)$ for all $m_X$-semiopen set $O$ of $(X, m_X)$.

(iii) $(m_X, m_Y)$-regular open if $f(O)$ is $m_Y$-regular open in $(Y, m_Y)$ for every $m_X$-open set $O$ of $(X, m_X)$.

Observation 3.1. The following Lemma 3.1 generalize the Theorem 4.2, given in [1].

Lemma 3.1. Let $(X, m_X)$ and $(Y, m_Y)$ be two m-spaces where $m_X$ satisfies the property of Maki. If $f : (X, m_X) \mapsto (Y, m_Y)$ is $\pi$-$(m_X, m_Y)$-irresolute function and $(m_X, m_Y)$-pre semiclosed, then $f(A)$ is $m_Y$-$\pi$-gs-closed for every $m_X$-$\pi$-gs-closed set $A$ in $X$.

Proof. Let $A$ be any $m_X$-$\pi$-gs-closed set in $X$, $U$ an $m_Y$-$\pi$-closed set in $Y$ such that $f(A) \subseteq U$. By hypothesis $f^{-1}(U)$ is $m_X$-open set in $X$ and $A \subseteq f^{-1}(U)$. Follows that $m_X$-$sCl(A) \subseteq f^{-1}(U)$, in consequence $f(m_X$-$sCl(A)) \subseteq U$. Since $A \subseteq m_X$-$sCl(A)$, then $f(A) \subseteq f(m_X$-$sCl(A))$, in consequence, $m_Y$-$sCl(f(A)) \subseteq m_Y$-$sCl(f(m_X$-$sCl(A)))$. Since $f$ is $(m_X, m_Y)$-pre semiclosed, $m_Y$-$sCl(f(m_X$-$sCl(A))) = f(m_X$-$sCl(A))$. Follows that $m_Y$-$sCl(f(A)) \subseteq f(m_X$-$sCl(A)) \subseteq U$. In consequence $f(A)$ is $m_Y$-$\pi$gs-closed set in $Y$.

Observation 3.2. The following Lemma 3.2 generalize the Theorem 4.3, given in [1].

Lemma 3.2. Let $(X, m_X)$ and $(Y, m_Y)$ be two m-spaces, where $m_Y$ satisfies the property of Maki. If $f : (X, m_X) \rightarrow (Y, m_Y)$ is $(m_X, m_Y)$-irresolute, $(m_X, m_Y)$-regular open and bijective, then $f$ is $\pi$-gs-$(m_X, m_Y)$-irresolute.
Proof. Let $F$ any $m_{Y}$-$\pi$-gs-closed set in $Y$ and $U$ any $m_{X}$-$\pi$-open set in $X$ such that $f^{-1}(F) \subseteq U$. Follows that $F \subseteq f(U)$ since $f(U)$ is $m_{Y}$-$\pi$-open, then $m_{Y}$-$sCl(F) \subseteq f(U)$, therefore $f^{-1}(m_{Y}$-$sCl(F)) \subseteq U$. Since $f$ is $(m_{X}, m_{Y})$-irresolute, then $f^{-1}(m_{Y}$-$sCl(F))$ is $m_{X}$-semiclosed, in consequence $m_{X}$-$sCl(f^{-1}(F)) \subseteq m_{X}$-$sCl(f^{-1}(m_{Y}$-$sCl(F))) = (f^{-1}(m_{Y}$-$sCl(F))) \subseteq U$. Follows that $f^{-1}(F)$ is $m_{X}$-$\pi$-gs-closed in $X$. \hfill $\square$

Lemma 3.3. Let $(X, m_{X})$ and $(Y, m_{Y})$ be two $m$-spaces where $m_{Y}$ satisfies the property of Maki. The following conditions are equivalent:

(i) $f : (X, m_{X}) \rightarrow (Y, m_{Y})$ is $(m_{X}, m_{Y})$-irresolute function.
(ii) For each subset $A \subseteq X$, $f(m_{X}$-$sCl(A)) \subseteq m_{Y}$-$sCl(f(A))$.
(iii) For each $m_{Y}$ semiclosed subset $V \subseteq Y$, the inverse image $f^{-1}(V)$ is an $m_{X}$ semiclosed in $X$.
(iv) For all $B \subseteq Y$, $m_{X}$-$sCl(f^{-1}(B)) \subseteq f^{-1}(m_{Y}$-$sCl(B))$.

Proof. (iii) $\Rightarrow$ (ii): Let $A$ be a subset of $X$ and suppose that $y \notin m_{Y}$-$sCl(f(A))$, then there exists a $m_{Y}$-semi open set $G$ in $Y$, such that $y \in G$ and $f(A) \cap G = \emptyset$, therefore, $f^{-1}(f(A) \cap G) = \emptyset$, it says that $A \cap f^{-1}(G) = \emptyset$. In consequence, $m_{X}$-$sCl(A) \subseteq f^{-1}(f^{-1}(G)) = \emptyset$, follows that $f(m_{X}$-$sCl(A)) \cap G = \emptyset$; and therefore, $y \notin f(m_{X}$-$sCl(A))$. But it is said that $f(m_{X}$-$sCl(A)) \subseteq m_{Y}$-$sCl(f(A))$ for all subset $A$ of $X$.

(ii) $\Rightarrow$ (iii): Let $V$ any $m_{Y}$-semiclosed subset in $Y$, then $f^{-1}(V) \subseteq X$. By hypothesis $f(m_{X}$-$sCl(f^{-1}(V))) \subseteq m_{Y}$-$sCl(f(f^{-1}(V)))$, follows that $f(m_{X}$-$sCl(f^{-1}(V))) \subseteq m_{Y}$-$sCl(V)$. In consequence, $f(m_{X}$-$sCl(f^{-1}(V))) \subseteq V$, follows that $m_{X}$-$sCl(f^{-1}(V) \subseteq f^{-1}(V)$. Therefore $f^{-1}(V)$ is an $m_{X}$-semiclosed set.

(ii) $\Rightarrow$ (iv): Let $B$ be a subset of $Y$, then $f^{-1}(B) \subseteq X$. Using the hypothesis, that $f(m_{X}$-$sCl(f^{-1}(B))) \subseteq m_{Y}$-$sCl(f(f^{-1}(B))) \subseteq m_{Y}$-$sCl(B)$, therefore, $m_{X}$-$sCl(f^{-1}(B)) \subseteq f^{-1}(m_{Y}$-$sCl(B))$.

(iv) $\Rightarrow$ (iii): Suppose that $V$ is an $m_{Y}$-semiclosed set in $Y$. Then $f^{-1}(V) \subseteq X$, by hypothesis, we obtain that $m_{X}$-$sCl(f^{-1}(V)) \subseteq f^{-1}(m_{Y}$-$sCl(V))$. But $V$ is a $m_{Y}$-semiclosed set, then $m_{X}$-$sCl(V) = V$. In consequence, $m_{X}$-$sCl(f^{-1}(V)) \subseteq f^{-1}(V)$. But this says that $f^{-1}(V)$ is an $m_{X}$-semiclosed set in $X$.

The others implications (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i), follow from the definition of $(m_{X}, m_{Y})$-irresolute function and the complement of set. \hfill $\square$

There are some relation between irresoluteness and continuity as we shows:

Theorem 3.4. Let $f : (X, m_{X}) \rightarrow (Y, m_{Y})$, then:

(i) If $f$ is $\pi$-$g$-$(m_{X}, m_{Y})$-irresolute, then $f$ is $\pi$-$g$-$(m_{X}, m_{Y})$-continuous.
(ii) If $f$ is $\pi$-$gs$-$(m_{X}, m_{Y})$-irresolute, then $f$ is $\pi$-$gs$-$(m_{X}, m_{Y})$-continuous.

1. If $f$ is $\pi$-$gp$-$(m_{X}, m_{Y})$-irresolute, then $f$ is $\pi$-$gp$-$(m_{X}, m_{Y})$-continuous.

and none of them are reversible.

Proof. The proof follows from Theorem 2.6 and 2.7. \hfill $\square$
The following example shows that there exists $\pi-(m_X, m_Y)$-irresolute maps but does not is $\pi-(m_X, m_Y)$-continuous.

**Example 3.10.** Let $X = Y = \{a, b, c, d\}$, $m_X = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{c, d\}\}$, and $m_Y = \{\emptyset, Y, \{a\}, \{b\}, \{a, c\}\}$. Then the $m_X$-regular open sets of $(X, m_X)$ are $\emptyset$, $X$, $\{a\}$, $\{b\}$, $\{a, b\}$, $\{c, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, and $\{a, b, c\}$. Define a function $f : (X, m_X) \mapsto (Y, m_Y)$ as: $f(a) = f(d) = d$, $f(b) = a$ and $f(c) = c$. Then $f$ is $\pi-(m_X, m_Y)$-irresolute but not $\pi-(m_X, m_Y)$-continuous.