On le-Γ-Semigroups

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Abstract

All the results of poe-groupoids, poe-semigroups (and in particular of \(\vee e\)-semigroups and le-semigroups) based on ideal elements can be transferred into ordered \(\Gamma\)-groupoids, ordered \(\Gamma\)-semigroups. This paper serves as an example to show the way we pass from poe-semigroups to poe-\(\Gamma\)-semigroups, from le-semigroups to le-\(\Gamma\)-semigroups. Besides, the results on poe- and le-\(\Gamma\)-semigroups, and in particular the results of the present paper, can be also obtained as application of the corresponding results on ordered semigroups. Independently, one can prove them in the way indicated in this paper.

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1. Introduction

We wrote this paper in an attempt to show the way we pass from ordered poe-groupoids, poe-semigroups, \(\vee e\)-semigroups and le-semigroups to poe-\(\Gamma\)-groupoids, poe-\(\Gamma\)-semigroups, \(\vee e\)-\(\Gamma\)-semigroups and le-\(\Gamma\)-semigroups and to emphasize the fact that all the results on the above mentioned ordered semigroups can be transferred in a natural way to the corresponding results on ordered \(\Gamma\)-semigroups.

The concept of a \(\Gamma\)-semigroup has been introduced by M. K. Sen in 1981 as follows: A nonempty set \(S\) is called a \(\Gamma\)-semigroup if the following assertions are satisfied: (1) \(aab \in S\) and \(aa\beta \in \Gamma\) and (2) \((aal\beta)c = a(al\beta)c = aa(b\beta)c\) for all \(a, b, c \in S\) and all \(a, b, c \in \Gamma\) [2]. In 1986, M. K. Sen and N. K. Saha changed that definition considering the following more general definition: Given two nonempty sets \(M\) and \(\Gamma\), \(M\) is called a \(\Gamma\)-semigroup if (1) \(aab \in M\) and (2)
\((aob)\beta c = a\alpha(b\beta c)\) for all \(a, b, c \in M\) and all \(\alpha, \beta \in \Gamma\) \[3\]. One can find that definition of \(\Gamma\)-semigroups in \[6\], where the notion of radical in \(\Gamma\)-semigroups and the notion of \(\Gamma S\)-act over a \(\Gamma\)-semigroup has been introduced, in \[4\] and \[5\], where the notions of regular and orthodox \(\Gamma\)-semigroups have been introduced and studied. Later, in \[1\], Saha calls a nonempty set \(S\) a \(\Gamma\)-semigroup if there is a mapping \(S \times \Gamma \times S \to S\) \(((a\gamma b) \to a\gamma b\) such that \((a\alpha b)\beta c = a\alpha(b\beta c)\) for all \(a, b, c \in S\) and all \(\alpha, \beta \in \Gamma\), and remarks that most usual semigroup concepts, in particular regular and inverse \(\Gamma\)-semigroups, have their analogous in \(\Gamma\)-semigroups. Although his first definition with Sen, where \(\Gamma\) plays the role of binary relations was better than his second one given by means of a mapping, the uniqueness condition was missing from the first one. Many authors tried to transfer results of semigroups or ordered semigroups to \(\Gamma\)-semigroups or ordered \(\Gamma\)-semigroups (shortly \(po\)-\(\Gamma\)-semigroups) some of them using the definition of a \(\Gamma\)-semigroup introduced by Sen in 1981, others the second one given by Sen in 1986. The main reason that they used the two definitions is that in an expression of the form \(a\gamma b\mu c\nu d\rho e\), for example, where \(a, b, c, d, e \in M\) and \(\gamma, \mu, \nu, \rho \in \Gamma\), it was not clear were to put the parenthesis. On the other hand, it is more convenient, of course, to define the \(\Gamma\)-semigroup \(M\) via a set \(\Gamma\) of binary relations than to define it as a mapping of \(M \times \Gamma \times M\) into \(M\). Adding the uniqueness condition in the definition given by Sen and Saha in \[3\], we avoid to define it via mappings.

2. Main Results

For two nonempty sets \(M\) and \(\Gamma\), define \(MTM\) as the set of all elements of the form \(m_1\gamma m_2\), where \(m_1, m_2 \in M\), \(\gamma \in \Gamma\). That is,

\[MTM := \{m_1\gamma m_2 \mid m_1, m_2 \in M, \gamma \in \Gamma\}.\]

**Definition 1.** Let \(M\) and \(\Gamma\) be two nonempty sets. The set \(M\) is called a \(\Gamma\)-**groupoid** if the following assertions are satisfied:

1. \(MTM \subseteq M\).
2. If \(m_1, m_2, m_3, m_4 \in M\), \(\gamma_1, \gamma_2 \in \Gamma\) such that \(m_1 = m_3\), \(\gamma_1 = \gamma_2\) and \(m_2 = m_4\), then \(m_1\gamma_1 m_2 = m_3\gamma_2 m_4\).

(\(\Gamma\) is a set of binary operations on \(M\)).

\(M\) is called a \(\Gamma\)-**semigroup** if, in addition, the following assertion is satisfied:

3. \((m_1\gamma_1 m_2)\gamma_3 m_3 = m_1\gamma_1(m_2\gamma_2 m_3)\) for all \(m_1, m_2, m_3 \in M\) and all \(\gamma_1, \gamma_2 \in \Gamma\).

In other words, \(M\) is a \(\Gamma\)-semigroup if \(\Gamma\) is a set of binary operations on \(M\) and \((m_1\gamma_1 m_2)\gamma_3 m_3 = m_1\gamma_1(m_2\gamma_2 m_3)\) for all \(m_1, m_2, m_3 \in M\) and all \(\gamma_1, \gamma_2 \in \Gamma\).

According to that ”associativity” relation, each of the elements \((m_1\gamma_1 m_2)\gamma_3 m_3\), and \(m_1\gamma_1(m_2\gamma_2 m_3)\) is denoted as \(m_1\gamma_1 m_2\gamma_2 m_3\).

That is,
Using conditions (1)–(3) one can prove that for an element of the form 
$m_1γ_1m_2γ_2m_3γ_3m_4$ one can put a parenthesis in any expression beginning with 
some $m_i$ and ending in some $m_j$, that is,

$$m_1γ_1m_2γ_2m_3γ_3m_4 := (m_1γ_1m_2)γ_2m_3γ_3m_4 = m_1γ_1(m_2γ_2m_3)γ_3m_4 = m_1γ_1(m_2γ_2m_3γ_3m_4) = (m_1γ_1m_2)m_2γ_2m_3γ_3m_4.$$

In general, for any element of the form

$$m_1γ_1m_2γ_2m_3γ_3m_4 \ldots \gamma_{n-1}m_nγ_nm_{n+1},$$

one can put a parenthesis in any expression beginning with some $m_i$ and ending in some $m_j$, that is, for example, in

$$m_1γ_1(m_2γ_2m_3)γ_3m_4 \ldots \gamma_{n-1}m_nm_{n+1}$$

There are several examples of $\Gamma$-semigroups in the bibliography. However, the example below based on Definition 1 above, shows clearly what a $\Gamma$-semigroup is.

**Example 2.** Consider the set $M := \{a, b, c, d\}$, and let $\Gamma = \{γ, μ\}$ be the set of two binary operations on $M$ defined in the tables below:

<table>
<thead>
<tr>
<th>$γ$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$a$</td>
<td>$b$</td>
<td>$c$</td>
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</tr>
<tr>
<td>$b$</td>
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<td>$a$</td>
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<td>$c$</td>
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<td>$d$</td>
<td>$d$</td>
<td>$a$</td>
<td>$b$</td>
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</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$μ$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
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<tr>
<td>$a$</td>
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<td>$c$</td>
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</tbody>
</table>

Since $(xρy)ωz = xρ(yωz)$ for all $x, y, z \in M$ and all $ρ, ω \in Γ$, $M$ is a $Γ$-

semigroup.

An ordered $Γ$-groupoid (shortly $po-Γ$-groupoid) (resp. $po-Γ$-semigroup) defined by Sen and Seth in [7], is a $Γ$-groupoid (resp. $Γ$-semigroup) $M$ together with an order relation $≤$ on $M$ such that $a ≤ b$ implies $aγc ≤ bγc$ and $cγa ≤ cγb$ for all $c \in M$ and all $γ \in Γ$. A $poe-Γ$-groupoid is a $Γ$-groupoid possessing a greatest element $e$. Let $M$ be a $poe-Γ$-groupoid and $γ \in Γ$. An element $a$ of $M$ is called $γ$-idempotent if $aγa = a$ [8], and it is called $γ$-subidempotent if $aγa ≤ a$. An element $a$ of $M$ is called a $γ$-right ideal element if $aγe ≤ a$. It is called a $γ$-left ideal element if $eγa ≤ a$. An element $q$ of $M$ is called a $γ$-quasi-ideal element if $aγe ∧ eγa$ exists and $aγe ∧ eγa ≤ a$. In particular, if $M$ is a $poe-Γ$-semigroup, then an element $b$ of $M$ is called a $γ$-bi-ideal element if $bγe ≤ b$ [9]. An element $a$ of a $poe-Γ$-semigroup $M$ is called $γ$-regular if $a ≤ aγeγa$. It is called $γ$-intra-regular if $a ≤ eγaγaγe$ [9]. A $poe-Γ$-semigroup
\( M \) is called \( \gamma\)-regular (resp. \( \gamma\)-intra-regular) if every element of \( M \) is so. A \( \forall e-\Gamma\)-semigroup is a semigroup at the same time an upper semilattice such that \((a \lor b)\gamma c = a\gamma c \lor b\gamma c \) and \(a\gamma(b \lor c) = a\gamma b \lor a\gamma c \) for all \(a, b, c \in M\) and all \(\gamma \in \Gamma\). In particular, if \( M \) is a lattice, then it is called an \( le-\Gamma\)-semigroup. Clearly, each \( \forall e-\Gamma\)-semigroup is a \( poe-\Gamma\)-semigroup.

**Lemma 3.** Let \( M \) be a \( poe-\Gamma\)-semigroup. If \( M \) is \( \gamma\)-regular, then the \( \gamma\)-right and the \( \gamma\)-left ideal elements of \( M \) are \( \gamma\)-idempotent.

**Proof.** Let \( a \) be a \( \gamma\)-right ideal element of \( M \). Since \( M \) is \( \gamma\)-regular, we have \( a \leq (a\gamma e)\gamma a \leq a\gamma a \leq a\gamma e \leq a \). If \( b \) is a \( \gamma\)-left ideal element of \( M \), then \( b \leq b\gamma(e\gamma b) \leq b\gamma b \leq e\gamma b \leq b \).

**Lemma 4.** Let \( M \) be a \( \gamma\)-regular \( poe-\Gamma\)-semigroup, \( a \) (resp. \( b \)) a \( \gamma\)-right (resp. \( \gamma\)-left) ideal element of \( M \) and \( x \in M \) such that \( a \land x \) (resp. \( x \land b \)) exists. Then \( a \land x \leq a\gamma x \) (resp. \( x \land b \leq x\gamma b \)).

**Proof.** Let \( a \) be a \( \gamma\)-right ideal element of \( M \) and \( x \in M \) such that \( a \land x \) exists. Since \( M \) is \( \gamma\)-regular, we have \( a \land x \leq (a \land x)\gamma\gamma(a \land x) \leq (a\gamma e)\gamma x \leq a\gamma x \). Let now \( b \) be a \( \gamma\)-left ideal element of \( M \) and \( x \in M \) such that \( a \land x \) exists. Then \( x \land b \leq (x \land b)\gamma\gamma(x \land b) \leq x\gamma(e\gamma b) \leq x\gamma b \).

**Lemma 5.** If \( M \) is a \( \gamma\)-regular \( poe-\Gamma\)-semigroup, a \( \gamma\)-right and \( b \) a \( \gamma\)-left ideal element of \( M \) such that \( a \land b \) exists, then \( a \land b = a\gamma b \).

**Proof.** By Lemma 4, \( a \land b \leq a\gamma b \). On the other hand, \( a\gamma b \leq a\gamma e \leq a \) and \( a\gamma b \leq e\gamma b \leq b \), so \( a\gamma b \leq a \land b \). Thus we have \( a \land b = a\gamma b \).

**Theorem 6.** An \( le-\Gamma\)-semigroup \( M \) is \( \gamma\)-regular if and only if for every \( \gamma\)-right ideal element \( a \) and every \( \gamma\)-left ideal element \( b \) of \( M \), we have \( a \land b \leq a\gamma b \) (equivalently \( a \land b = a\gamma b \)).

**Proof.** \( \implies \). It follows from Lemma 5.

\( \impliedby \). Let \( a \in M \). Since the element \( a\gamma e \lor a \) (resp. \( e\gamma b \lor b \)) is a right (resp. left) ideal element of \( M \), by hypothesis, we have

\[
\begin{align*}
a & \leq (a\gamma e \lor a) \land (e\gamma a \lor a) \\
& \leq (a\gamma e \lor a)\gamma(e\gamma a \lor a) \\
& = a\gamma(e\gamma e)a \lor a\gamma e a \lor a\gamma a = a\gamma e a \lor a\gamma a
\end{align*}
\]

Then \( a\gamma a \leq (a\gamma e a \lor a\gamma a)\gamma a = a\gamma(e\gamma e)a \lor a\gamma a \leq a\gamma e a \). Then \( a \leq a\gamma e a \), and \( M \) is \( \gamma\)-regular.

**Lemma 7.** Let \( M \) be a \( \gamma\)-regular \( poe-\Gamma\)-semigroup, \( x \) a \( \gamma\)-right and \( y \) a \( \gamma\)-left ideal element of \( M \) such that \( x \land y \) and \( (x \land y)\gamma e \land e\gamma(x \land y) \) exist. Then the element \( x\gamma y \) is a \( \gamma\)-quasi-ideal element of \( M \).

**Proof.** We have \( (x \land y)\gamma e \land e\gamma(x \land y) \leq x e \gamma e \leq x \) and \( (x \land y)\gamma e \land e\gamma(x \land y) \leq e\gamma y \leq y \). Then \( (x \land y)\gamma e \land e\gamma(x \land y) \leq x \land y \), and \( x \land y \) is a \( \gamma\)-quasi-ideal
element of \( M \). On the other hand, by Lemma 5, \( x \land y = x \gamma y \), so \( x \gamma y \) is a \( \gamma \)-quasi-ideal element of \( M \).

**Theorem 8.** An \( le-\Gamma \)-semigroup \( M \) is \( \gamma \)-regular if and only if the \( \gamma \)-right ideal elements and the \( \gamma \)-left ideal elements of \( M \) are \( \gamma \)-idempotent, and for every \( \gamma \)-right ideal element \( x \) and every \( \gamma \)-left ideal element \( y \) of \( M \), the element \( x \gamma y \) is a \( \gamma \)-quasi-ideal element of \( M \).

**Proof.** \( \implies \). It follows by Lemma 3 and Lemma 7.

\( \Longleftarrow \). Let \( x \in M \). Since \( x \gamma e \lor x \) is a \( \gamma \)-right ideal element of \( M \), it is \( \gamma \)-idempotent, so we have
\[
x \leq x \gamma e \lor x = (x \gamma e \lor x) \gamma (x \gamma e \lor x) = x \gamma (e \gamma x \lor x) \lor x \gamma (e \gamma x) \lor x \gamma x \leq x \gamma e.
\]

Since \( e \gamma x \lor x \) is a \( \gamma \)-right ideal element of \( M \), in a similar way, we get \( x \leq e \gamma x \). Hence we obtain \( x \leq x \gamma e \land e \gamma x \).

As \( e \) is a \( \gamma \)-right ideal element of \( M \), it is \( \gamma \)-idempotent, so \( x \gamma e \gamma x = x \gamma (e \gamma x) \gamma x = (x \gamma e) \gamma (e \gamma x) \gamma x \). By hypothesis, \( (x \gamma e) \gamma (e \gamma x) \) is a \( \gamma \)-quasi-ideal element of \( M \), then \( x \gamma e \gamma x \) is so i.e. \( (x \gamma e) \gamma e \land e \gamma (x \gamma e) \gamma x \leq x \gamma e \gamma x \). Hence we obtain \( x \leq x \gamma e \gamma x \), and \( M \) is \( \gamma \)-regular.

It might be noted that \( (x \gamma e) \gamma e \land e \gamma (x \gamma e) \gamma x = x \gamma e \gamma x \). Indeed, since \( e \gamma x \) and \( x \gamma e \) are \( \gamma \)-idempotent, we have \( x \gamma (e \gamma x) = x \gamma (e \gamma x) \gamma (e \gamma x) = x \gamma e \gamma x \gamma (e \gamma x) \leq x \gamma e \gamma x \gamma e \) and \( (x \gamma e) \gamma x = (x \gamma e) \gamma (x \gamma e) \gamma x = (x \gamma e) \gamma x \gamma e \gamma x \leq e \gamma x \gamma e \gamma x \), thus we have \( x \gamma e \gamma x \leq (x \gamma e) \gamma e \land e \gamma (x \gamma e) \gamma x \).

**Lemma 9.** Let \( M \) be a \( poe-\Gamma \)-semigroup, \( x \) a \( \gamma \)-right ideal element of \( M \) and \( y \in M \). Then \( x \gamma y \) is a \( \gamma \)-bi-ideal element of \( M \).

**Proof.** In fact,
\[
(x \gamma y) \gamma e \gamma (x \gamma y) = x \gamma (y \gamma e) \gamma x \gamma y \leq (x \gamma e) \gamma x \gamma y \\
\leq x \gamma x \gamma y \leq (x \gamma e) \gamma y \leq x \gamma y.
\]

In a similar way the following holds true

**Lemma 10.** Let \( M \) be a \( poe-\Gamma \)-semigroup, \( y \) a \( \gamma \)-left ideal element of \( M \) and \( x \in M \). Then \( x \gamma y \) is a \( \gamma \)-bi-ideal element of \( M \).

**Lemma 11.** Let \( M \) be a \( poe-\Gamma \)-semigroup. If \( M \) is \( \gamma \)-regular, then for every \( \gamma \)-bi-ideal element \( b \) of \( M \), we have \( b = b \gamma e \gamma b \).

**Proof.** Since \( b \) is a \( \gamma \)-bi-ideal element of \( M \), \( b \gamma e \gamma b \leq b \). Since \( M \) is \( \gamma \)-regular, \( b \leq b \gamma e \gamma b \). Then \( b = b \gamma e \gamma b \).
Lemma 12. If M is a $\gamma$-regular poe-$\Gamma$-semigroup, then the $\gamma$-bi-ideal elements of M are $\gamma$-subidempotent.

Proof. Let $b$ be a $\gamma$-bi-ideal element of M. By Lemma 11, $b = b\gamma e\gamma b$. Then $b\gamma b = b\gamma (e\gamma b)\gamma b \leq b\gamma e\gamma b = b$. \hfill \Box

Theorem 13. Let M be a $\vee e\Gamma$-semigroup. If M is a $\gamma$-regular, then $b$ is a $\gamma$-bi-ideal element of M if and only if there exist a $\gamma$-right ideal element $x$ and a $\gamma$-left ideal element $y$ of M such that $b = x\gamma y$.

Proof. $\Longrightarrow$. Let M be $\gamma$-regular and $b$ a $\gamma$-bi-ideal element of M. The element $b\gamma e \vee b$ is a $\gamma$-right ideal element, the element $e\gamma b \vee b$ is a $\gamma$-left ideal element of M and

$$(b\gamma e \vee b)(e\gamma b \vee b) = b\gamma (e\gamma e)\gamma b \vee b\gamma e\gamma b \vee b\gamma b = b\gamma e\gamma b \vee b\gamma b.$$ 

On the other hand, by Lemma 12, $b\gamma b \leq b$ and, by Lemma 11, $b = b\gamma e\gamma b$. Thus we have $(b\gamma e \vee b)(e\gamma b \vee b) = b \vee b\gamma b = b$.

$\Longleftarrow$. If $x$ is a $\gamma$-right ideal element and $y$ a $\gamma$-left ideal element then, by Lemma 9 (or Lemma 10), $x\gamma y$ is a $\gamma$-bi-ideal element of M. \hfill \Box

Lemma 14. Let M be a $\gamma$-regular poe-$\Gamma$-semigroup at the same semilattice under $\wedge$ and $x_1, x_2$ (resp. $y_1, y_2$) $\gamma$-right (resp. $\gamma$-left) ideal elements of M. Then

$$x_1 \wedge x_2 \leq x_1\gamma x_2 \quad (\text{resp. } y_1 \wedge y_2 \leq y_1\gamma y_2).$$

Proof. First of all, $(x_1 \wedge x_2)\gamma (x_1 \wedge x_2) \leq x_1\gamma x_2$. Besides, as $x_1, x_2$ are $\gamma$-right ideal elements of M, $x_1 \wedge x_2$ is a $\gamma$-right ideal element of M, as well. As so, by Lemma 3, it is $\gamma$-idempotent. Hence we obtain $x_1 \wedge x_2 \leq x_1\gamma x_2$. The case for the $\gamma$-left ideal elements is similar. \hfill \Box

Lemma 15. Let M be a poe-$\Gamma$-semigroup, $x_1$ (resp. $y_1$) a $\gamma$-right (resp. $\gamma$-left) ideal element of M. Then, for each $x \in M$ (resp. $y \in M$), the element $x\gamma x_1$ (resp. $y_1\gamma y$) is a $\gamma$-right (resp. $\gamma$-left) ideal element of M.

Proof. Indeed, $(x\gamma x_1)\gamma e = x\gamma (x_1\gamma e) \leq x\gamma x_1$ and $e\gamma (y_1\gamma y) = (e\gamma y_1)\gamma y \leq y_1\gamma y$. \hfill \Box

Theorem 16. An le-$\Gamma$-semigroup M is $\gamma$-regular if and only if for every $\gamma$-right ideal element $x$, every $\gamma$-left ideal element $y$, and every $\gamma$-bi-ideal element $b$ of M, we have $x \wedge b \wedge y \leq x\gamma b\gamma y$.

Proof. $\Longrightarrow$. Let $x$ be a $\gamma$-right, $y$ a $\gamma$-left, and $b$ a $\gamma$-bi-ideal element of M. Since M is $\gamma$-regular, by Theorem 13 and Lemma 5, there exist a $\gamma$-right ideal element $x_1$ and a $\gamma$-left ideal element $y_1$ of M such that $b = x_1\gamma y_1 = x_1 \wedge y_1$. Then $x \wedge b \wedge y = x \wedge x_1 \wedge y_1 \wedge y$. Since $x$ is a $\gamma$-right and $y$ a $\gamma$-left ideal element of M, by Lemma 4, we have $x \wedge x_1 \leq x\gamma x_1$ and $y_1 \wedge y \leq y_1\gamma y$. Hence we have $x \wedge b \wedge y \leq x\gamma x_1 \wedge y_1\gamma y$. As $x\gamma x_1$ (resp. $y_1\gamma y$) is a $\gamma$-right (resp. $\gamma$-left) ideal element of M, by Theorem 6, we obtain
\[ x\gamma x_1 \land y_1 \gamma y = (x\gamma x_1)\gamma (y_1 \gamma y) = x\gamma (x_1 \gamma y_1) \gamma y = x \gamma b \gamma y. \]

Thus we have \( x \land b \land y \leq x \gamma b \gamma y. \)

\( \Leftarrow \). Let \( x \) be a \( \gamma \)-right and \( y \) a \( \gamma \)-left ideal element of \( M \). The element \( e \) is a \( \gamma \)-bi-ideal element of \( M \), because \( e \gamma (e \gamma e) \leq e \gamma e \leq e \). By hypothesis, we have \( x \land y = x \land e \land y \leq (e \gamma y) \gamma y \leq x \gamma y \). Then, by Theorem 6, \( M \) is \( \gamma \)-regular. \( \square \)

**Remark 17.** The necessary condition of Theorem 16 can be also proved as follows: We remark first, that the property

\[ a \leq a \gamma e \gamma a \gamma e a \]

characterizes the \( \gamma \)-regular \( po-\Gamma \)-semigroups. In fact, if \( M \) is \( \gamma \)-regular then, for each \( a \in M \), we have \( a \leq a \gamma e a \leq (a \gamma e a) \gamma e (a \gamma e a) = a \gamma e a \gamma e a \). Conversely, if \( a \leq a \gamma e a \gamma e a \gamma e a \) for each \( a \in M \), then for each \( a \in M \), we have

\[ a \leq a \gamma (e \gamma a) \gamma (e \gamma a) \gamma e a \leq a \gamma e a \gamma e a = a \gamma (e \gamma e) \gamma e a \leq a \gamma e a, \]

and \( M \) is \( \gamma \)-regular.

\( \Rightarrow \). Let now \( x \) be a \( \gamma \)-right ideal element, \( y \) a \( \gamma \)-left ideal element, and \( b \) a \( \gamma \)-bi-ideal of \( M \). Since \( M \) is \( \gamma \)-regular, we have

\[ x \land b \land y \leq (x \land b \land y) \gamma e (x \land b \land y) \gamma e (x \land b \land y) \gamma e (x \land b \land y) \gamma e (y \land b \land y) \gamma e (x \land y) \gamma (b \gamma e b) \gamma (e \gamma y) \leq x \gamma b \gamma y. \]

Analogous characterizations given in Theorem 16, one can find in the propositions below.

**Proposition 18.** A \( po-\Gamma \)-semigroup \( M \) is \( \gamma \)-intra-regular if and only if for every \( \gamma \)-right ideal element \( x \), every \( \gamma \)-left ideal element \( y \), and every \( \gamma \)-bi-ideal element \( b \) of \( M \), we have \( x \land b \land y \leq y \gamma b \gamma x \).

**Proposition 19.** A \( po-\Gamma \)-semigroup \( M \) is both \( \gamma \)-regular and \( \gamma \)-intra-regular if and only if for every \( \gamma \)-right ideal element \( x \), every \( \gamma \)-left ideal element \( y \), and every \( \gamma \)-bi-ideal element \( b \) of \( M \), we have \( x \land b \land y \leq b \gamma x \gamma y \).

**Proposition 20.** A \( po-\Gamma \)-semigroup \( M \) is \( \gamma \)-intra-regular and the \( \gamma \)-left ideal elements of \( M \) are \( \gamma \)-idempotent if and only if for every \( \gamma \)-right ideal element \( x \), every \( \gamma \)-left ideal element \( y \), and every \( \gamma \)-bi-ideal element \( b \) of \( M \), we have \( x \land b \land y \leq y \gamma x \gamma b \).

**Proposition 21.** A \( po-\Gamma \)-semigroup \( M \) is \( \gamma \)-intra-regular and the \( \gamma \)-right ideal elements of \( M \) are \( \gamma \)-idempotent if and only if for every \( \gamma \)-right ideal element \( x \), every \( \gamma \)-left ideal element \( y \), and every \( \gamma \)-bi-ideal element \( b \) of \( M \), we have \( x \land b \land y \leq b \gamma y \gamma x \).
Conclusion. If \((S, \cdot)\) is a semigroup, \(\gamma\) a symbol \((\gamma \notin S)\) and define \(a\gamma b := ab\) for all \(a, b \in S\), then \(S\) is a \(\{\gamma\}\)-semigroup. "Conversely" if \(M\) is a \(\Gamma\)-semigroup, take a \(\gamma \in \Gamma\) and define a multiplication on \(M\) as \(a.b := a\gamma b\), then \((M, \cdot)\) is a semigroup. Which means that if \(M\) is a \(\{\gamma\}\)-semigroup, then \((M, \cdot)\) is a semigroup.

References


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