Integral Transform Method in Some Mixed Problems

Naser A. Hoshan

Department of Mathematics,
Tafila Technical University, P.O. Box 179
Tafila, Jordan
drnaserh@ttu.edu.jo

Abstract

Paper is devoted to determine the solution of a none stationary heat conduction equation in axial symmetric cylindrical coordinates under mixed discontinuous boundary conditions of the third kind. The solution of the considered boundary value problem is obtained with the use of dual integral equations and reduced to integral equation of the second kind of Volterra type.

Keywords: Dual integral equations, mixed boundary conditions

Introduction

In this paper we examine some classical methods for a homogeneous heat conduction equation in axial symmetrical cylindrical coordinates under mixed discontinuous boundary conditions of the third kind acted on the surface of a semi-space of a solid infinite cylinder. A new kind of dual integral equations with time dependent were used to determine the solution of a given boundary problem with the use of integral transform and separation of variables. Discussion deals with a given mixed boundary value problem and dual integral equations in this paper which is reduced to Volterra type integral equation, based on earlier results were obtained in monographs [1,2,6,7].
Formulation of the problem

Find a temperature distribution function \( \theta =\theta(r,z,\tau) \), \( 0< r < \infty, 0< z < \infty \) initially at temperature zero with boundary conditions bounded at zero and infinity obeys a third kind mixed discontinuous boundary conditions

\[
\alpha_1\theta_z - \lambda_1\theta = -f_1(r,\tau), \quad r \in I
\]

\[
\alpha_2\theta_z - \lambda_2\theta = -f_2(r,\tau), \quad r \in \overline{I} \tag{1}
\]

\( \alpha_1, \lambda_1, i=1,2 \) are constants, \( I = \{0< r < R, z = 0\} \), \( \overline{I} = \{R< r < \infty, z = 0\} \), \( R \) is the line of discontinuity \( \theta_z = \partial \theta / \partial z \), \( f_i(r,\tau), i=1,2 \) known functions satisfy the properties

\[
\int_0^\infty [f_i'(r,\tau)] \sqrt{r}dr < \infty, \quad f_i'(r,\tau) = \frac{1}{2} [f_i(r+0,\tau) + f_i(r-0,\tau)], \quad \int_0^\infty [f_i'(r,\tau)dr < \infty .
\]

Applying a Laplace transform to the given boundary value problem with respect to \( \tau \) \( \theta(r,z,\tau) = L[\theta(r,z,\tau)] \), then separating variables to heat conduction equation in cylindrical coordinates, a general solution should be written in form of improper integral

\[
\theta(r,z,s) = \int_0^\infty A(p,s) J_0(pr) \exp(-z \sqrt{p^2 + k^2})dp \tag{3}
\]

\( A(p,s) \) unknown functions, \( J_0(pr) \) is a Bessel function of order zero of the first kind, \( k^2 = s/\alpha, \) \( s \) is a parameter of the Laplace transform, \( \text{Re}(s) > 0, \alpha \neq 0 \) is the temperature conductivity coefficient (constant).

Take a mixed boundary conditions (1) and (2) for expression (3), we get the following dual integral equations to determine the unknown function \( A(p,s) \) in the Laplace transform image

\[
\int_0^\infty A(p,s) J_0(pr) (\alpha_1 \sqrt{p^2 + k^2} + \lambda_1)dp = f_1(r,s), \quad r \in I \tag{4}
\]

\[
\int_0^\infty A(p,s) J_0(pr) (\alpha_2 \sqrt{p^2 + k^2} + \lambda_2)dp = f_2(r,s), \quad r \in \overline{I} \tag{5}
\]

Equations (4) and (5) should be written in a standard form

\[
\int_0^\infty B(p,s) J_0(pr) u(p,s)dp = f_1(r,s), \quad r \in I \tag{6}
\]

\[
\int_0^\infty B(p,s) J_0(pr) dp = f_2(r,s), \quad r \in \overline{I} \tag{7}
\]
Integral transform method

where

\[ u(p,s) = \left( \frac{\alpha_1 \sqrt{p^2 + k^2 + \lambda_1}}{\alpha_2 \sqrt{p^2 + k^2 + \lambda_2}} \right), \quad \lim_{p \to \infty} u(p,s) = \alpha_1/\alpha_2 = \gamma < \infty \]

\[ B(p,s) = A(p,s) \left( \frac{\alpha_2 \sqrt{p^2 + k^2 + \lambda_2}}{\alpha_2 \sqrt{p^2 + k^2 + \lambda_2}} \right), \]

\[ r(p,s) = u(p,s) - \gamma = \frac{\lambda_1 \alpha_2 - \lambda_2 \alpha_1}{\alpha_2 (\alpha_2 \sqrt{p^2 + k^2 + \lambda_2})}. \]

At \( s \to 0 \), the dual integral equations (6),(7) were reduced to known equations [9].

Rewrite (7) in form

\[ \int_0^\infty B(p,s) J_0(pr) dp = \begin{cases} \phi(r,s), & r \in I \\ f_2(r,s), & r \in \tilde{I}. \end{cases} \quad (8) \]

\( \phi(r,s) = L[\phi(r,\tau)] \) is unknown function. Applying to (8) an inverse Hankel integral transform [3] in the interval \( I \cup \tilde{I} \) we have

\[ B(p,s) = \int_0^\infty y p J_0(py) \phi(y,s) dy + \int_0^\infty y p J_0(py) f_2^\prime(y,s) dy \quad (9) \]

Substituting (9) into (6), then interchanging the order of integration, a Fredholm integral equation of the second kind is obtained for determination the unknown function \( \phi(r,s) \)

\[ \gamma \phi(r,s) + \int_0^\infty \phi(y,s) K(r,y,s) dy = F(r,s), \quad r \in I \quad (10) \]

with kernel

\[ K(r,y,s) = \int_0^\infty p J_0(py) J_0(pr) r(p,s) dp \]

and free term

\[ F(r,s) = f_1(r,s) + \int_0^\infty \int_0^\infty y p f_2(r,s) J_0(pr) J_0(py) r(p,s) dp dy \]

the kernel and the free term should be satisfied [7]

\[ \int_0^\infty |F(r,s)| dr < \infty, \quad \int_0^\infty \int_0^\infty K^2(r,y,s) dr dy < \infty. \quad (11) \]

Now, applying to (10) an inverse Laplace transform, we get an integral equation of Volterra type

\[ \gamma \phi(r,\tau) + \int_\Omega K(r,y,\tau - \xi) \phi(r,\xi) d\xi dy = F(r,\tau), \quad (12) \]

\( \Omega = 0 < \xi < \tau, 0 < y < R \). \( K(r,y,\tau) = L^{-1} K(r,y,s) \), \( F(r,\tau) = L^{-1} F(r,s) \), for more details about the inverse Laplace transform see for example [3].

Integral equation (12) play important role in theory of integral equations and partial differential equation of parabolic type [4], it can be solved numerically with the help of some mathematical software packages such as Mathematica or Matlab [5,8].
References


Received: March, 2009