

Delay-Dependent Exponential Stability of Linear Systems with Fast Time-Varying Delay

Le Van Hien

Department of Mathematics
Hanoi National University of Education
136 Xuan Thuy, Hanoi, Vietnam
Hienlv@hnue.edu.vn

Abstract

This paper presents new exponential stability conditions for a class of linear systems with time-varying delay. Unlike the previous works, the delay function considered in this paper is time-varying without any requirement on the derivative, which means that a fast time-varying delay is allowed. Based on an improved Lyapunov-Krasovskii functional combined with Newton-Leibniz formula and free-weighting matrix method, the exponential stability conditions are derived in terms of Kharitonov-type linear matrix inequalities (LMIs), which allows one to compute simultaneously the two bounds that characterize the exponential stability rate of the solution.

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1 Introduction

During the last decade, considerable attention has been devoted to the problem of stability analysis for time delay systems due to theoretical and practical importance. There is a great number of stability results, where the Lyapunov-Krasovskii functional method is certainly used as main tool [2, 4, 13]. These results can be classified into two types according to their dependence of the delay size, that is, delay-dependent conditions and delay-independent conditions. Because delay-dependent criteria make use of information on the delay size, they are less conservative than the delay-independent ones. For systems with time-varying delay, to derive delay-dependent conditions, it is usually assumed that the delay function is differentiable and upper bounded [3, 6, 14, 16, 17].

While most papers provided conditions for asymptotic stability of time delay systems, it is interesting to find the transient decay rate of delay system ([5,7-13,15]). Recently, based on linear matrix inequalities approach [1], a systematic procedure for finding exponential stability conditions has been proposed in [8] for LTI systems with constant delay. In [5, 15], by using state transformation $\xi(t) = e^{\lambda t}x(t)$, delay-dependent conditions for robust exponential stability of linear uncertain systems with constant delay were given in terms of LMIs. By also using the state transformation method, [12] gives conditions for the exponential stability of non-autonomous systems with constant delay in terms of solution of Riccati-type differential equation. However, in many cases, the delays may not be differentiable even not be continuous, how to get delay-dependent exponential stability conditions is still of interest.

In this paper, we consider the exponential stability problem for linear time-varying systems with fast time varying delay. To reduce the conservatism of the stability conditions, an improved Lyapunov-Krasovskii functional combined with Newton-Leibniz formula is introduced. The distributed delay free-weighting matrix functional combined with the Newton-Leibniz formula avoids the restriction on the derivative of time-varying delay. Moreover, the conditions obtained in this paper are also formulated in terms of LMIs [8], which can be efficiently solved by using various convex optimization algorithm [1].

The paper is organized as follows. After Introduction, in Section 2 we give notations, definitions and technical lemmas needed for the proof of the main result. Sufficient conditions for the exponential stability of the systems and an illustrated example are presented in Section 3. The paper ends with conclusions and cited references.

2 Preliminary Notes

The following notations will be used throughout this paper: R^+ denotes the set of all real non-negative numbers; R^n denotes the n -dimensional space with the scalar product $\langle \cdot, \cdot \rangle$ and the Euclidean norm $\|\cdot\|$; A^T denotes the transpose of A ; I denotes the identity matrix; $\lambda(A)$ denotes the set of all eigenvalues of A ; $\lambda_{\max}(A) = \max\{\text{Re } \lambda : \lambda \in \lambda(A)\}$; $\lambda_{\min}(A) = \min\{\text{Re } \lambda : \lambda \in \lambda(A)\}$; $\|A\|$ denotes the spectral norm of matrix A , that is $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$; $C([a, b], R^n)$ denotes the set of all R^n -valued continuous functions on $[a, b]$; A matrix A is semi-positive definite ($A \geq 0$) if $\langle Ax, x \rangle \geq 0$, for all $x \in R^n$; A is positive definite ($A > 0$) if $\langle Ax, x \rangle > 0$ for all $x \neq 0$; $A \geq B$ means $A - B \geq 0$.

Consider a linear system with time-varying delay of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1x(t - \tau(t)), \quad t \in R^+, \\ x(t) &= \phi(t), \quad t \in [-h, 0], \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state; A, A_1 are given real matrices and $\phi \in C([-h, 0], R^n)$ is the initial function with the norm $\|\phi\| = \sup_{s \in [-h, 0]} \|\phi(s)\|$; $\tau(t)$ is the time-varying delay function satisfying $0 \leq \tau(t) \leq h$. This means that the upper bound for the time varying delay is available.

Remark 2.1. *The time delay function $\tau(t)$ considered in this paper is time varying, may be unknown and no restriction on the existence of derivative of $\tau(t)$ is needed, which allows the time delay to be a fast time varying function.*

Definition 2.2. *Given $\alpha > 0$. The system (1) is α -exponentially stable if there exists a number $\gamma \geq 1$ such that every solution $x(t, \phi)$ of the system satisfies the following condition*

$$\|x(t, \phi)\| \leq \gamma e^{-\alpha t} \|\phi\|, \quad t \in R^+.$$

We need the following facts and lemma to use in the proof of our main result.

Fact 1.([1]) For any matrices X, Y, Z , where $X = X^T, Y = Y^T > 0$ then $X + Z^T Y^{-1} Z < 0$ if and only if

$$\begin{bmatrix} X & Z^T \\ Z & -Y \end{bmatrix} < 0, \quad \text{or} \quad \begin{bmatrix} -Y & Z \\ Z^T & X \end{bmatrix} < 0.$$

Fact 2.(Cauchy matrix inequality) For any $x, y \in R^n$ and symmetric positive definite matrix W , we have

$$2x^T y \leq x^T W x + y^T W^{-1} y.$$

Lemma 2.3. *For any symmetric matrix $Q > 0$ and any matrix F , the following inequality holds:*

$$-\int_{t-\tau(t)}^t \dot{x}^T(s) Q \dot{x}(s) ds \leq \xi^T(t) \tilde{F} \xi(t) + h \xi^T(t) \bar{F}^T Q^{-1} \bar{F} \xi(t),$$

where

$$\begin{aligned} \bar{F} &= [0 \quad 0 \quad F], \\ \tilde{F} &= \text{diag}\{0, 0, F + F^T\}, \xi^T(t) = \left[x^T(t) \quad \dot{x}^T(t) \quad \int_{t-\tau(t)}^t \dot{x}^T(s) ds \right]. \end{aligned}$$

Proof. Utilizing fact 2, we have

$$-2\dot{x}^T(s) \bar{F} \xi(t) \leq \dot{x}^T(s) Q \dot{x}(s) + \xi^T(t) \bar{F}^T Q^{-1} \bar{F} \xi(t),$$

which implies

$$-\int_{t-\tau(t)}^t \dot{x}^T(s) Q \dot{x}(s) ds \leq 2 \left(\int_{t-\tau(t)}^t \dot{x}(s) ds \right)^T \bar{F} \xi(t) + \int_{t-\tau(t)}^t \xi^T(t) \bar{F}^T Q^{-1} \bar{F} \xi(t) ds$$

$$\begin{aligned} &\leq 2\xi^T(t) \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \bar{F}\xi(t) + \tau(t)\xi^T(t)\bar{F}^T Q^{-1}\bar{F}\xi(t) \\ &\leq \xi^T(t)\tilde{F}\xi(t) + h\xi^T(t)\bar{F}^T Q^{-1}\bar{F}\xi(t). \end{aligned}$$

□

3 Main results

In this section, the exponential stability of the system (1) is analyzed. Sufficient conditions for delay-dependent exponential stability of the system are derived in terms of Kharitonov-type linear matrix inequality [8] by using an improved Lyapunov-Krasovskii functional combined with Newton-Leibniz formula and free weighting matrix method.

The main result is stated as follow.

Theorem 3.1. *Given $\alpha > 0$. The system (1) is α -exponentially stable if there exist symmetric positive definite matrices P, Q and matrices F, N, N_1, N_2 satisfying the following LMI:*

$$\begin{bmatrix} \Omega & \mu\bar{F}^T \\ \mu\bar{F} & -e^{2\alpha h}Q \end{bmatrix} < 0, \tag{2}$$

Moreover, the solution $x(t, \phi)$ of the system satisfies

$$\|x(t, \phi)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} e^{-\alpha t} \|\phi\|, \quad t \in R^+, \tag{3}$$

where

$$\begin{aligned} \Omega &= \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ * & \Omega_{22} & \Omega_{23} \\ * & * & \Omega_{33} \end{bmatrix}, \\ \Omega_{11} &= \bar{A}^T P + P\bar{A} + 2\alpha P + N\bar{A} + \bar{A}^T N^T, \\ \Omega_{12} &= \bar{A}^T N_1^T - N, \\ \Omega_{13} &= \bar{A}^T N_2^T - N A_1 - P A_1, \\ \Omega_{22} &= hQ - N_1 - N_1^T, \\ \Omega_{23} &= -N_2^T - N_1 A_1, \\ \Omega_{33} &= e^{-2\alpha h}(F + F^T) - A_1^T N_2^T - N_2 A_1, \\ \bar{A} &= A + A_1, \quad \mu = \sqrt{h}, \quad \bar{F} = \begin{bmatrix} 0 & 0 & F \end{bmatrix}, \\ \alpha_1 &= \lambda_{\min}(P), \alpha_2 = \lambda_{\max}(P) + h^2 \lambda_{\max}(Q) [\lambda_{\max}(A^T A) + \lambda_{\max}(A_1^T A_1)] \end{aligned}$$

Proof. Supplementing the definition of $\phi(t)$ on the interval $[-h - \tau_0, 0]$, where $\tau_0 = \tau(0)$, as

$$\bar{\phi}(t) = \begin{cases} \phi(t), & t \in [-h, 0], \\ \phi(-h), & t \in [-h - \tau_0, -h] \end{cases},$$

and that of $\dot{x}(t)$ on the interval $[-h, 0]$ as

$$\dot{x}(t) = A\bar{\phi}(t) + A_1\bar{\phi}(t - \tau_0), t \in [-h, 0].$$

By using Newton - Leibniz formula we have

$$x(t - \tau(t)) = x(t) - \int_{t-\tau(t)}^t \dot{x}(s)ds, \quad t \in R^+.$$

Then system (1) can be rewritten as

$$\dot{x}(t) = \bar{A}x(t) - A_1 \int_{t-\tau(t)}^t \dot{x}(s)ds, \quad t \in R^+. \tag{4}$$

The process of obtaining (4) from (1) is sometime known as model transformation. The stability of (4) implies that of (1) although the stability of the two systems expressed by (1) and (4) are not equivalent .

Consider the following Lyapunov-Krasovskii functional

$$V(x_t) = x^T(t)Px(t) + \int_{-h}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} \dot{x}^T(\theta)Q\dot{x}(\theta)d\theta ds \tag{5}$$

It is easy to verify that

$$V(x_t) \geq \alpha_1 \|x(t)\|^2, \quad t \in R^+, \tag{6}$$

where $\alpha_1 = \lambda_{\min}(P)$.

Taking derivative of V along trajectory of system (4) we obtain

$$\begin{aligned} \dot{V}(x_t) &= x^T(t) [\bar{A}^T P + P\bar{A}] x(t) - 2x^T(t)PA_1 \int_{t-\tau(t)}^t \dot{x}(s)ds \\ &\quad + h\dot{x}^T(t)Q\dot{x}(t) - \int_{-h}^0 e^{2\alpha s} \dot{x}^T(t+s)Q\dot{x}(t+s)ds - 2\alpha V(x_t) \\ &\leq x^T(t) [\bar{A}^T P + P\bar{A}] x(t) - 2x^T(t)PA_1 \int_{t-\tau(t)}^t \dot{x}(s)ds \\ &\quad + h\dot{x}^T(t)Q\dot{x}(t) - e^{-2\alpha h} \int_{t-\tau(t)}^t \dot{x}^T(s)Q\dot{x}(s)ds - 2\alpha V(x_t). \end{aligned}$$

Therefore,

$$\begin{aligned} \dot{V}(x_t) + 2\alpha V(x_t) &\leq x^T(t) [\bar{A}^T P + P\bar{A} + 2\alpha P] x(t) \\ &\quad - 2x^T(t) P A_1 \int_{t-\tau(t)}^t \dot{x}(s) ds \\ &\quad + h\dot{x}^T(t) Q \dot{x}(t) - e^{-2\alpha h} \int_{t-\tau(t)}^t \dot{x}^T(s) Q \dot{x}(s) ds. \end{aligned} \tag{7}$$

Applying Lemma 2.3, we have

$$-e^{-2\alpha h} \int_{t-\tau(t)}^t \dot{x}^T(s) Q \dot{x}(s) ds \leq e^{-2\alpha h} \xi^T(t) \tilde{F} \xi(t) + h e^{-2\alpha h} \xi^T(t) \bar{F}^T Q^{-1} \bar{F} \xi(t). \tag{8}$$

To get a less conservative stability condition, we add the following zero equation with any matrices N, N_1, N_2

$$2 \left[x^T(t) N + \dot{x}^T(t) N_1 + \left(\int_{t-\tau(t)}^t \dot{x}(s) ds \right) N_2 \right] \times \left[\bar{A} x(t) - \dot{x}(t) - A_1 \int_{t-\tau(t)}^t \dot{x}(s) ds \right] = 0,$$

which can be represented as

$$\xi^T(t) M \xi(t) = 0,$$

where

$$M = \begin{bmatrix} \bar{A}^T N^T + N\bar{A} & \bar{A}^T N_1^T - N & \bar{A}^T N_2^T - N A_1 \\ * & -N_1^T - N_1 & -N_2^T - N_1 A_1 \\ * & * & -A_1^T N_2^T - N_2 A_1 \end{bmatrix}. \tag{9}$$

Combining (7), (8) and (9) we get

$$\dot{V}(x_t) + 2\alpha V(x_t) \leq \xi^T(t) [\Omega + h e^{-2\alpha h} \bar{F}^T Q^{-1} \bar{F}] \xi(t) \tag{10}$$

Using Schur complement [1] for $X = \Omega, Z = \mu \bar{F}$ and $Y = -e^{2\alpha h} Q$, inequality (2) implies that

$$\Omega + \mu^2 e^{-2\alpha h} \bar{F}^T Q^{-1} \bar{F} < 0$$

Therefore, from (10) we finally obtain

$$\dot{V}(x_t) + 2\alpha V(x_t) \leq 0, \quad \forall t \in R^+,$$

which yields

$$V(x_t) \leq V(\phi) e^{-2\alpha t}, t \in R^+.$$

Next, we will estimate $V(\phi)$ from (5) as follow

$$\begin{aligned} V(\phi) &\leq \lambda_{\max}(P)\|\phi\|^2 + \lambda_{\max}(Q) \int_{-h}^0 \int_s^0 e^{2\alpha\tau} \|\dot{x}(\theta)\|^2 d\theta ds \\ &\leq \lambda_{\max}(P)\|\phi\|^2 + \lambda_{\max}(Q) \int_{-h}^0 \int_s^0 \|A\phi(\theta) + A_1\bar{\phi}(\theta - \tau_0)\|^2 d\theta ds \\ &\leq \lambda_{\max}(P)\|\phi\|^2 + 2\lambda_{\max}(Q) [\|A\|^2 + \|A_1\|^2] \|\phi\|^2 \int_{-h}^0 \int_s^0 d\theta ds \\ &= [\lambda_{\max}(P) + h^2\lambda_{\max}(Q) (\lambda_{\max}(A^T A) + \lambda_{\max}(A_1^T A_1))] \|\phi\|^2. \end{aligned}$$

Furthermore, taking (6) into account, we obtain

$$\alpha_1\|x(t, \phi)\|^2 \leq V(x_t) \leq \alpha_2 e^{-2\alpha t} \|\phi\|^2.$$

Therefore,

$$\|x(t, \phi)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} e^{-\alpha t} \|\phi\|, t \in R^+,$$

which concludes the proof. □

Remark 3.2. For given $\alpha > 0$, by iteratively solving the LMI given in Theorem 3.1 with respect to h , one can find the maximum upper bound of time delay function that guarantees exponential stability of system (1) with decay rate α .

Remark 3.3. By using the similar method proposed in Theorem 3.1, we can also get exponential stability condition for systems with multiple time delays.

Numerical Example.

Consider the linear time-varying delay system (1), where

$$\begin{aligned} \tau(t) &= \begin{cases} 1 & \text{if } t \in Q^+, \\ 0 & \text{if } t \in R^+ \setminus Q^+, \end{cases} \\ A_0 &= \begin{bmatrix} -1.5 & 0 \\ 0.1 & 0.1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.5 & 0 \\ -0.1 & -0.7 \end{bmatrix}. \end{aligned}$$

It's worth noting that the delay function $\tau(t)$ is nowhere differentiable even not continuous function on R^+ . And also, both matrices A and A_1 are unstable.

For this system, the criteria for exponential stability obtained in [5, 7, 8, 12, 13, 15] are not applicable. Taking $\alpha = 0.25$, then LMI (2) is satisfied with

$$\begin{aligned} P &= \begin{bmatrix} 2.3056 & 0.1518 \\ 0.1518 & 0.6348 \end{bmatrix}, \\ Q &= I, N = N_2 = 0, N_1 = I, F = -I. \end{aligned}$$

By Theorem 3.1, the system is exponentially stable with decay rate $\alpha = 0.25$. We derive from Theorem 3.1 the stability factor $\gamma = 2.4829$ and the solution $x(t, \phi)$ satisfies

$$\|x(t, \phi)\| \leq 2.4829 e^{-0.25t} \|\phi\|, \quad t \in R^+.$$

4 Conclusions

Based on an improved Lyapunov-Krasovskii functional combined with Newton-Leibniz formula and free weighting matrix method, we have presented new sufficient conditions for the exponential stability of linear systems with time-varying delay. The conditions have been formulated in terms of Kharitonov-type linear matrix inequality, which allows to compute simultaneously the two bounds that characterize the exponential stability rate of the solution. Numerical example showing the effectiveness of our condition is given.

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