When Every Projective Ideal is Finitely Generated

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Abstract

In this paper we introduce and investigate a class of those rings in which every projective ideal is finitely generated. We establish the transfer of this notion to trivial ring extensions, pullbacks, and direct products; and then generate new and original families of rings satisfying this property.

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1 Introduction

Throughout this paper all rings are assumed to be commutative with identity element. Let \( R \) be a ring and let \( M \) be an \( R \)-module. As usual, we use \( \text{pd}_R(M) \) and \( \text{fd}_R(M) \) to denote the usual projective and flat dimensions of \( M \), respectively. If \( R \) is an integral domain, we denote its quotient field by \( qf(R) \).

In this paper, we are interested to those rings in which every projective ideal is finitely generated and which will be called \( PFG \)-rings. In particular, local rings, integral domains rings, and Noetherien rings are examples of \( PFG \)-rings.

Let \( A \) be a ring and \( E \) an \( A \)-module. The trivial ring extension of \( A \) by \( E \) (also called the idealization of \( E \) over \( A \)) is the ring \( R := A \times E \) whose underlying group is \( A \times E \) with multiplication given by \( (a, e)(a', e') = (aa', ae' + a'e) \). For the reader’s convenience, recall that if \( I \) is an ideal of \( A \) and \( E' \) is a submodule of \( E \) such that \( IE \subseteq E' \), then \( J := I \times E' \) is an ideal
of $R$. Suitable background on commutative trivial ring extensions is [3, 4, 5].

The purpose of this paper is to study the transfer of the PFG notion to the trivial ring extensions, pullbacks, and direct products. In this line, we provide a new family of examples of non-local, non-coherent and non-integral domain PFG-rings.

2 Main Results

First, we explore trivial ring extensions of the form $R := D \propto E$, where $D$ is an integral domain and $E$ is a $K(= qf(D))$-vector space. Notice in this context that $(a, b) \in R$ is regular if and only if $a \neq 0$. The main result (Theorem 2.1) examines the transfer of PFG-property to $R$ and hence generates new examples of PFG-rings.

**Theorem 2.1** Let $D$ be a domain, $K := qf(D)$, $E$ be a $K$-vector space and let $R := D \propto E$ be the trivial ring extension of $D$ by $E$. Then $R$ is an PFG-ring.

We need the following Lemma before proving Theorem 2.1.

**Lemma 2.2** Let $T := K \propto E$ be the trivial ring extension of a field $K$ by a $K$-vector space $E$. Then there exists no proper projective ideal of $T$. In particular, $T$ is an PFG-ring.

**Proof** Let $J := 0 \propto E'$ be a proper ideal of $T$, where $E'(\subseteq E)$ is a $K$-vector space. We claim that $J$ is not projective. Deny. Then $J$ is a free $T$-module since $T$ is a local ring and so $J$ is a principal ideal generated by a regular element, a contradiction since $J(0, e) = 0_T$ for each $e \in T$. Hence, $J$ is not projective, as desired.

**Proof of Theorem 2.1.** Let $J$ be a nonzero projective ideal of $R$. Set $T := K \propto E$ which is a flat $R$-module since $T = S^{-1}R$, where $S = D - \{0\}$. Hence, $JT(= J \otimes_R T)$ is a nonzero projective ideal of $T$ and so $JT = T = K \propto E$ by Lemma 2.2. Therefore, there exists $(a, e) \in J$ such that $a \neq 0$ which implies that $J = I \propto E = I \otimes_D R$ for some nonzero ideal $I$ of $D$. We claim that $I$ is a projective ideal of $D$.

For any $D$-module $N$, we have by [2, p.118],

$$Ext_D(I, N \otimes_D R) \cong Ext_R(I \otimes_D R, N \otimes_D R) = 0$$
On the other hand, \( N \) is a direct summand of \( N \otimes_D R \) since \( D \) is a direct summand of \( R \). Therefore, \( \text{Ext}_D(I,N) = 0 \) for all \( D \)-module \( N \). This means that \( I \) is a projective ideal of \( D \).

Therefore, \( I \) is a finitely generated \( D \)-module and then \( J := I \otimes_D R \) is a finitely generated ideal of \( R \) and this completes the proof of Theorem 2.1.

We recall that a coherent ring is a ring such that each finitely generated ideal is finitely presented. In particular, any Noetherian ring is a coherent ring. See for instance [3].

Theorem 2.1 enriches the literature with new examples of non-local and non-Noetherian \( PFG \)-rings with zerodivisors, as shown below.

**Example 2.3** Let \( Z \) be the ring of integers, \( Q = qf(Z) \) and let \( R := Z \otimes Q \).

1) \( R \) is an \( PFG \)-ring by Theorem 2.1.
2) \( R \) is not local by [4, Theorem 25.1(3)] since \( Z \) is not local.
3) \( R \) is not coherent by [5, Theorem 2.8(1)]. In particular, \( R \) is not Noetherian.

Now, we study the transfer of the \( PFG \)-property to pullbacks. These constructions have been proven to be useful in solving many open problems and conjectures in various contexts in ring theory. See for instance [1, 3, 6].

**Theorem 2.4** Let \( A \subseteq B(:= S^{-1}A) \) be an extension of rings, where \( S \) is a multiplicative subset of \( A \), and \( Q \) is an ideal of both \( A \) and \( B \). Assume that \( B \) is a local ring. Then \( A \) is an \( PFG \)-ring provided \( A/Q \) is an \( PFG \)-ring.

We need the following Lemma before proving Theorem 2.4.

**Lemma 2.5** Let \( A, B, S \) and \( Q \) be as in Theorem 2.4. Assume that \( B \) is a local ring and let \( I \) be a projective ideal of \( A \). Then there exists \( 0 \neq x \in B \) and an ideal \( I' \supseteq Q \) of \( A \) such that \( I \otimes A/Q \cong I'/Q \) as \( A/Q \)-modules and \( I = xI' \cong I' \) as \( A \)-modules.

**Proof** Let \( I \) be a projective ideal of \( A \). We have \( I \otimes A/Q \cong I/IQ \). Since \( IB(\cong I \otimes_A B) \) is a projective ideal of \( B \) and since \( B \) is a local ring, \( IB \) is a free ideal of \( B \), hence principal ideal of \( B \). Then there exists \( 0 \neq x \in B \) such that \( IB = xB \); so \( IQ = IQB = Q(IB) = xBQ = xQ \). Also, by replacing \( x \) with a suitable \( x' \), we may assume without loss of generality that \( I = xI' \), where \( I' \) is an ideal of \( A \). (In detail: \( \forall i = 1, \ldots, m \), we have \( a_i \in I \subseteq IB = xB \), then
∃b_i ∈ A and ∃s_i ∈ S such that a_i = x(b_i/s_i). Thus, for x' = x/\prod_{j=1}^{m} s_j ∈ B, we have a_i = x'b_i', where b'_i = (\prod_{j=1,j\neq i}^{m} s_j)b_i ∈ A and I' = \sum_{i=1}^{m} Ab'_i. Then I = x'I'; and IB = xB = x'B since elements of S are units in B.) Therefore, IQ = xQ and I = xI' \cong I' as A-modules, where I' is an ideal of A, so that we have: I \otimes A/Q \cong I/IQ = xI'/xQ \cong I'/Q as A/Q-modules.

**Proof of Theorem 2.4.** Let A ⊆ B(:= S^{-1}A) be an extension of rings, where S is a multiplicative subset of A, Q is an ideal of both A and B, and assume that B is a local ring. Assume that A/Q is an PFG-ring and let I be a projective ideal of A. Then I \otimes_A B := IB is a principal (free) ideal by Lemma 2.5. On the other hand, I \otimes_A (A/Q) \cong I'/Q is a projective ideal of A/Q by Lemma 2.5 and so is finitely generated since A/Q is an PFG-ring. Therefore, I is a finitely generated ideal by [3, Theorem 5.1.1(3)] and this completes the proof of Theorem 2.4.

Theorem 2.4 enriches the literature with new examples of non-local and non-Noetherian PFG-rings with zerodivisors, as shown below.

**Example 2.6** Let D be a non-local integral domain, K := qf(D), T := K[X]/(X^n) = K + M, where X is an indeterminate over K, n is a positive integer, and M = XT is a maximal ideal of a local ring T. Then:
1) R := D + M is an PFG-ring.
2) R is not local since D is not local.
3) R is not Noetherian since D is not Noetherian and R is a faithfully flat D-module.

Next, we study the transfer of the PFG-property to direct products.

**Proposition 2.7** Let (R_i)_{i=1,...,n} be a family of rings. Then, \prod_{i=1}^{n} R_i is an PFG-ring if and only if R_i is an PFG-ring for each i = 1, ..., n.

We need the following Lemma before proving Proposition 2.7.

**Lemma 2.8** ([6, Lemma 2.5]) Let (R_i)_{i=1,2} be a family of rings and E_i an R_i-module for i = 1, 2. Then:
1) E_1 \prod E_2 is a finitely generated R_1 \prod R_2-module if and only if E_i is a finitely generated R_i-module for i = 1, 2.
2) $E_1 \prod E_2$ is a projectif $R_1 \prod R_2$-module if and only if $E_i$ is a projectif $R_i$-module for $i = 1, 2$.

**Proof of Proposition 2.7.** By induction on $n$, it suffices to prove the assertion for $n = 2$. Since an ideal of $R_1 \prod R_2$ is of the form $I_1 \prod I_2$, where $I_i$ is an ideal of $R_i$ for $i = 1, 2$, and the conclusion follows easily from Lemma 2.8.

Proposition 2.7 enriches the literature with new examples of non-local and non-Noetherian $PFG$-rings with zerodivisors, as shown below.

**Example 2.9** Let $R_1$ be a Noetherien ring, $R_2$ be a non-Noetherien local ring, and let $R = R_1 \prod R_2$. Then:
1) $R$ is an $PFG$-ring by Proposition 2.7 since $R_i$ is an $PFG$-ring for each $i = 1, \ldots, 2$.
2) $R$ is not a local ring.
3) $R$ is not Noetherian since $R_2$ is not Noetherian.

**References**


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