International Mathematical Forum, 4, 2009, no. 40, 1997 - 2000

A Formula for Solving a Special Case of Euler-Cauchy ODE

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Abstract

A quick and simple Formula has been introduced to solve a special case of non-homogenous n^{th} order Euler-Cauchy ODE.

Mathematics Subject Classification code: 65L99

Keywords: Euler-Cauchy ODE

1 Introduction

Our work here is very simple and can be understood easily by undergraduate students taking the first course in ordinary differential equations (ODEs), see [2]; therefore these results are not new and can be obtained by any good undergraduate student taking the first course in ODE. But I thought that such derived formulas might be useful especially for some problems in Engineering, Mathematics and Physics.

2 Preliminary Notes

Usually, to solve non-homogenous linear Euler-Cauchy Ordinary Differential Equations (ODEs), it is better to use the Method of Variation of Parameters since these kinds of ODEs are of non-constant coefficients. But in this very special case we found that the Method of Undetermined Coefficients may be used effectively noticing that the particular solution must be assumed as $y_p = Bx^m$. The following formula is very useful and saves a lot of time and a lot of calculations especially when n is very large.

3 Main Results

These are the main results of the paper.

1. A special case of a non-homogenous linear Euler-Cauchy ODE.

Let us consider a special case of the nth order non-homogenous linear Euler-Cauchy ODE; namely:

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + ... + a_1 x y^{'} + a_0 y = b x^m; \tag{1}$$

where $m \geq n, m$ must not equal any of the roots of the characteristic equation, and both m & n are integers.

Usually, to evaluate the particular solution of a non-homogenous Euler-Cauchy ODE, it is advisable to use the method of variation of parameters; but when n gets very large; mainly when $n \geq 4$, the calculations gets very messy if one does not use a mathematical software. Therefore, for few special cases of Euler-Cauchy ODE it is appropriate to use the method of undetermined coefficients.

But even when using the method of undetermined coefficients,with large n, the calculations , by hand, of course still huge but less messy than using the method of variation of parameters. Therefore I thought of deriving a very straight forward and simple Formula to help our undergraduate students solve such ODE in a very quick and short way as done in the next following section.

1.1 The Formula.

By direct substitution after assuming that the particular solution of (1) takes the form $y_p = Bx^m$; then using the method of undetermined coefficients we obtain the value of B to be as follows for $m \geq n$:

$$B = \frac{b}{m! \sum_{k=0}^{n} \frac{a_k}{(m-k)!}} \tag{2}$$

and thus the particular solution of (1) will be:

$$y_p = \frac{bx^m}{m! \sum_{k=0}^{n} \frac{a_k}{(m-k)!}} \tag{3}$$

Remark 1.1:

In general, other forms of $y_p = A_m x^m + A_{m-1} x^{m-1} + ... + A_1 x + A_0$ will not work since at least one of $A_0, A_1, ..., A_m$ will be arbitrary and thus y_p is not going to be unique which does not make sense.

1.2 Example.

Use (2) or (3) to solve:

$x^3y^{'''} - 3x^2y^{''} + 6xy^{'} - 6y = 5x^4$.
Solution:
Note that $n = 3, m = 4, b = 5, a_0 = -6, a_1 = 6, a_2 = -3, a_3 = 1$.
Using (2) we obtain:
$B = \frac{5}{4!\sum_{k=0}^{3}\frac{a_k}{(4-k)!}} = \frac{5}{4!(\frac{a_0}{4!}+\frac{a_1}{3!}+\frac{a_2}{2!}+\frac{a_3}{1!})} = \frac{5}{24(\frac{-6}{24}+\frac{6}{6}+\frac{-3}{2}+\frac{1}{1})} = \frac{5}{6}$
and thus by (3) we find that the particular solution will be:
$y_p = \frac{5}{6}x^4$.

2. General form of a non-homogenous linear n^{th} order Euler-Cauchy ODE.

$$a_nx^ny^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + ... + a_1xy^{'} + a_0y = r(x); \tag{4}$$

where $r(x) \neq 0$.

We shall now convert equation (4) to a constant coefficient ODE by using the substitution:

$$x = e^u \quad \text{or equivalently} \quad u = \ln x \tag{5}$$

As a result, by simple calculations, and letting $D = \frac{d}{du}$ it will be obvious, see [1], to show that (4) can be written as:

$$[a_nD(D-1)(D-2)...(D-n+1)+a_{n-1}D(D-1)(D-2)...(D-n+2)+...+a_1D+a_0]y = r(e^u) \tag{6}$$

Observe that equation (4) is a non-homogenous linear ODE with constant coefficients and, as known, it may be solved easily by finding the homogenous solution using the characteristic equation and the particular solution using either the Undetermined Coefficient or Variation of Parameters Methods; then the general solution of (6) will be the sum of both the homogenous and particular solutions.

2.1.1 Example.
Solve

$x^2y^{''} + 5xy^{'} + 3y = \ln x$ by converting it to a constant coefficient ODE then solve it.

Solution:

Let $u = \ln x$; then our equation will be converted to the equation:

$[D(D-1)+5D+3]y = u$; where $D = \frac{d}{du}$. Which in turns gives the general solution:

$y = c_1e^{-3u} + c_2e^{-u} + \frac{1}{3}u - \frac{4}{9}$; but since $u = \ln x$, we get the general solution to our equation as:

$y = c_1x^{-3} + c_2x^{-1} + \frac{1}{3}\ln x - \frac{4}{9}$.

2.1.1 Remark.

By using the above substitution; namely $u = \ln x$ or $x = e^u$ we can rewrite equation (1) in the form:

$$[a_nD(D-1)(D-2)...(D-n+1)+a_{n-1}D(D-1)(D-2)...(D-n+2)+...+a_1D+a_0]y = be^{mu} \tag{7}$$

where $D = \frac{d}{du}$ and $m \geq n$.

Of course, equation (7) becomes a constant coefficient non-homogenous linear ODE that can be easily solved by

regular known methods. And its particular solution will coincide with our solution obtained in (3).

References

[1] Frank Ayres , Jr., *Schaum's Outline of Differential Equations,* McGraw-Hill Book Company, 1952.

[2] Kreyszig, E., *Advanced Engineering Mathematics, 9th Edition,* John Wiley & Sons, INC., 2006.

Received: April, 2009