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Existence and Uniqueness of Solutions to a Long Range Diffusive Predator-Prey Model

Marwan S. Abualrub

Department of Mathematics, University of Jordan
P.O. Box 11942, Amman, Jordan
abualrubms@yahoo.com

Abstract

A model for predator-prey has been considered, existence and uniqueness to long range diffusion of such model has been shown.

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1 Introduction

Predator-prey models and competing species relations have been initiated by Lotka in [7],[8], and Volterra [9], in a classical way , to represent the interaction between populations in terms of simultaneous nonlinear differential equations.

Predator-prey systems have been described properly using mathematical models. Such models describe the interaction between multi-species, particularly two species for which the growth rate of one of the species is increased but decreased for the other.

In this paper, we include diffusion to a model that describes the interaction between the prey and the predator; and we use a time-continuous model rather than discrete, even though the problem is inherently discrete, since we expect a continuous overlap of generations in the population of such models. Therefore, continuous models might be a reasonable approximation.

In addition, we also include long range (or non-local) diffusion; and we shall consider the more realistic case; namely a diffusion model in the (x_1, x_2) plane instead of the one dimensional case. In such model, we consider two coupled nonlinear parabolic partial differential equations (PDEs) including diffusion; for more details see [5].

Later on, we will talk about the existence and uniqueness of solutions to our model.

In [3], a short range (or local) diffusion model has been considered and a traveling wave solution has been obtained to that model.

2 Preliminary Notes

1.The Predator- Prey Model

The model were are going to consider here is a modification to the planktonic patchiness model , a kind of predator-prey models, which was originally considered by Levin and Segel in 1976 , see [6];in which they considered some biological hypotheses concerning the origin of planktonic patchiness that let to their model in [6].

We used the model in [6] and include long range diffusion, then we assume that the species specific diffusion coefficients be constants to come up with the long range diffusive predator-prey model:

$$u_t - \Delta^{(2)} u = au + bu^2 - cuv + d\Delta u \qquad (1)^*$$

and

$$v_t - \Delta^{(2)} v = euv - fv^2 + g\Delta v \qquad (2)^*;$$

where $u = u(x,t)$ is the prey density, $v = v(x,t)$ is the predator density, $x = (x_1, x_2)$, $u_t = \frac{\partial u}{\partial t}$, and $v_t = \frac{\partial v}{\partial t}$.

We shall assume that a, b, c, d, e, f and g are constants however , a may assumed to be compact supported and bounded function of x (i.e., $a = a(x)$; $a(x) = 0$ for $|x| > N$; where N is a constant) not a constant. The reason for this assumption is that the birth (or death) rate may depend on the environment, which is assumed to be bounded. Another reason for assuming that a, b, c, d, e, f and g are constants, is due to the fact that the birth (or death) rate could just depend on the interaction between the male and the female (sexual interaction as in terms bu^2 and $-fv^2$) , or just on the binary interaction (sexual, birth or death like in the term uv).

Here $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ represents the short range diffusion (dispersal), and

$\Delta^{(2)} = \sum_{i,j=1}^{2} \frac{\partial^4}{\partial x_i^2 \partial x_j^2}$ represents the long range diffusion.

uv represents the binary interaction which result in either growth or decay.

au represents the birth or death of the prey.

u^2, v^2 represent the sexual interaction between the males and the females of the prey and the predator, respectively.

Finally, we shall assume that the initial data are in the same L^pspace for some $p > 1$. The initial values for equations $(1)^*$ and $(2)^*$ will be given by $u(x, 0) = g(x)$ and $v(x, 0) = h(x)$, respectively, where both $g(x)$ and $h(x) \in L^p(R^2)$.

In addition, we will consider large values of time t ,since we are looking for non-local solutions in the long range diffusion; as considered in [2].

In previous work; namely [1] we have shown the existence and uniqueness of solutions to the following short range diffusive model. We shall repeat the process here in order to enable the reader to know the difference between the short range and (later) the long range diffusion.

The model considered in [1] was as follows:

$$u_t - \Delta u = au + ku^2 - buv \tag{1}$$

and

$$v_t - \Delta v = cuv - dv^2 \tag{2}$$

The solution to the above model , as in [1], is redone again in the next section.

2. Usual or local or short range diffusion with $p = q$ in the $L^{p,q}$ norms

To ease solving (1) and (2) we shall make the term au disappear from (1); to do so let $u(x,t) = e^{\alpha t} w(x,t)$. Therefore (1) and (2) together with the initial conditions become as follows:

$$w_t - \Delta w = ke^{\alpha t} w^2 - bwv \tag{3}$$

$$w(x, 0) = g(x) \tag{4}$$

and

$$v_t - \Delta v = ce^{\alpha t} wv - dv^2 \tag{5}$$

$$v(x, 0) = h(x) \tag{6}$$

Since we have the heat operator $\frac{\partial}{\partial t} - \Delta$ in the left hand sides of (3) and (5), therefore w and v can be obtained by solving the following integral equations:

$$w = \int_0^t \int_{R^n} K(x-y, t-\tau) \left[ke^{\alpha\tau} w^2 - bwv\right] dyd\tau + \int_{R^n} K(x-y,t)g(y)dy \tag{7}$$

and

$$v = \int_0^t \int_{R^n} K(x-y, t-\tau) \left[ce^{\alpha\tau} wv - dv^2\right] dyd\tau + \int_{R^n} K(x-y,t)h(y)dy; \tag{8}$$

where K is the fundamental solution to the heat equation with $n = 2$;thus

$$K(x,t) = \frac{1}{2\pi t} e^{\frac{-|x|^2}{2t}} \text{ and } |x| = (x_1^2 + x_2^2)^{1/2} \tag{9}$$

Using the symbol $\otimes$ to represent the convolution in space and time while the symbol $*$ to represent the convolution in space only; we can rewrite (7) and (8) in a simpler way as follows:

$$w = K \otimes \left[ke^{\alpha\tau} w^2 - bwv\right] + K * g \tag{10}$$

$$v = K \otimes \left[ce^{\alpha\tau} wv - dv^2\right] + K * h; \tag{11}$$

where w and v are weak solutions of (3), (4), (5), and (6) respectively.

We will now introduce the function φ as follows:

$$\varphi(\tau) = \begin{cases} 0, & if \quad \tau > T^{'} \\ e^{\alpha\tau}, & if \quad \tau \leq T^{'} \end{cases} \tag{12}$$

where $0 < \tau < T^{'}$ and $T^{'}$ is small.

Let us now consider $T(w)$ and $T(v)$ to be the images of w and v respectively.. Thus, if $\tau \leq T^{'}$, then equations (10) and (11) become:

$$T(w) = K \otimes \left[k\varphi(\tau) w^2 - bwv\right] + K * g \tag{13}$$

$$T(v) = K \otimes \left[c\varphi(\tau) wv - dv^2\right] + K * h \tag{14}$$

We are assuming small values of t in order to show the existence and uniqueness of local solutions to (13), (14).

Lemma 2.1 *If* $w(x,t)$, $v(x,t) \in L^{\left(\frac{4}{2-\epsilon}, \frac{4}{2-\epsilon}\right)}\left(R^2 x \left[0, T^{'}\right]\right)$; *and* $g(x)$, $h(x) \in L^{\frac{8}{3(2-\epsilon)}}(R^2)$,

then for $\epsilon > 0$, *we have*

$\|T(w)\|_{\frac{4}{2-\epsilon}, \frac{4}{2-\epsilon}} \leq C^{'}_{\frac{4}{2-\epsilon}} \|w\|^2_{\frac{4}{2-\epsilon}, \frac{4}{2-\epsilon}} + C^{''}_{\frac{4}{2-\epsilon}} \|w\|_{\frac{4}{2-\epsilon}, \frac{4}{2-\epsilon}} \|v\|_{\frac{4}{2-\epsilon}, \frac{4}{2-\epsilon}} + C_{\frac{8}{3(2-\epsilon)}} \|g\|_{\frac{8}{3(2-\epsilon)}}$

and

$\|T(v)\|_{\frac{4}{2-\epsilon}, \frac{4}{2-\epsilon}} \leq D^{'}_{\frac{4}{2-\epsilon}} \|w\|_{\frac{4}{2-\epsilon}, \frac{4}{2-\epsilon}} \|v\|_{\frac{4}{2-\epsilon}, \frac{4}{2-\epsilon}} + D^{''}_{\frac{4}{2-\epsilon}} \|v\|^2_{\frac{4}{2-\epsilon}, \frac{4}{2-\epsilon}} + D_{\frac{8}{3(2-\epsilon)}} \|h\|_{\frac{8}{3(2-\epsilon)}}.$

Proof:

It is obvious from (9) that

$$0 \leq |K(x,t)| \leq \frac{c_0}{\left(|x| + t^{1/2}\right)^2}; \tag{15}$$

where c_0 is some positive constant.

Observe that

$\left(|x|+t^{1/2}\right)^2=\left(|x|+t^{1/2}\right)^{2-\theta}\left(|x|+t^{1/2}\right)^{\theta}\geq|x|^{2-\theta}\,t^{\frac{\theta}{2}}$
for $t>0$ and $0<\theta<2$.
Thus

$$|K(x,t)|\leq c_0\frac{1}{|x|^{2-\theta}}\frac{1}{t^{\frac{\theta}{2}}} \tag{16}$$

Now, for very small positive $\epsilon>0$, we may rewrite (16) as

$$|K(x,t)|\leq c_0\frac{1}{|x|^{2-\theta}}\frac{1}{t^{\frac{\theta}{2}}}\frac{t^{\frac{\epsilon}{2}}}{t^{\frac{\epsilon}{2}}} \tag{17}$$

Using the assumption that $0<t<T^{'}$ and $T^{'}$ is very small, (17) may be expressed as

$$|K(x,t)|\leq\frac{c_0T^{'^{\frac{\epsilon}{2}}}}{|x|^{2-\theta}t^{\frac{\theta+\epsilon}{2}}} \tag{18}$$

Observe that $\frac{\theta+\epsilon}{2}=1-\frac{2-(\theta+\epsilon)}{2}$. The very small value of $\epsilon>0$ is chosen to satisfy $0<\theta+\epsilon<2$; thus (18) may be written as:

$$|K(x,t)|\leq c_0\frac{1}{|x|^{2-\theta}}\frac{T^{'^{\frac{\epsilon}{2}}}}{t^{1-\frac{2-(\theta+\epsilon)}{2}}} \tag{19}$$

By (13) and (19) we obtain:

$$|T(w(x,t))|\leq kc_0e^{\alpha T^{'}}T^{'^{\frac{\epsilon}{2}}}\int_0^{T^{'}}\int_{R^2}\frac{|w(y,\tau)|^2dyd\tau}{|x-y|^{2-\theta}|t-\tau|^{1-\frac{2-(\theta+\epsilon)}{2}}}$$
$$+bc_0T^{'^{\frac{\epsilon}{2}}}\int_0^{T^{'}}\int_{R^2}\frac{|w(y,\tau)||v(y,\tau)|dyd\tau}{|x-y|^{2-\theta}|t-\tau|^{1-\frac{2-(\theta+\epsilon)}{2}}}+|K*g| \tag{20}$$

Let us now assume that $g\in L^p(R^2)$, $wv\in L^{\frac{p}{2}}(R^2)$, and $w^2=w.w\in L^{\frac{p}{2}}(R^2)$. Let $r>1$ be chosen so that:

$$\frac{1}{r}=\frac{2}{p}-\frac{\theta}{2},\quad 2<p<\frac{4}{\theta} \tag{21}$$

Now, by taking the $L^r(R^2)$ norm of both sides and using the same idea as in Benedek-Panzone Potential Theorem , see [4] page 321, theorem 1, the first and second terms of the right hand side of (20) become:

$$|T(w(.,t))|_r\leq kc_0c_pT^{'^{\frac{\epsilon}{2}}}\int_0^{T^{'}}\frac{|w(.,\tau)|_p^2d\tau}{|t-\tau|^{1-\frac{2-(\theta+\epsilon)}{2}}}+bc_0c_pT^{'^{\frac{\epsilon}{2}}}\int_0^{T^{'}}\frac{|w(.,\tau)|_p|v(.,\tau)|_pd\tau}{|t-\tau|^{1-\frac{2-(\theta+\epsilon)}{2}}}$$
$$+\|K*g\,(.,t)\,\|_r \tag{22}$$

Now, if we let $\|w(.,\tau)\|_p^2 \in L^{\frac{q}{2}}(R^+)$, $\|w(.,\tau)\|_p . \|v(.,\tau)\|_p \in L^{\frac{q}{2}}(R^+)$, and $s > 1$ so that

$$\frac{1}{s} = \frac{2}{q} - \frac{2-(\theta+\epsilon)}{2}\ ,\ 2 < q < \frac{4}{2-(\theta+\epsilon)} \tag{23}$$

Again, by applying the Benedek-Panzone Potential Theorem to the first and second terms of the right hand side of (22) , and upon taking the $L^s(R^+)$ norm of both sides, we obtain:

$$|T(w)|_{r,s} \leq kc_0c_pc_q|w|_{p,q}^2 + bc_0c_pc_q|w|_{p,q}|v|_{p,q} + |K*g|_{r,s} \tag{24}$$

We shall now take $p = r = \frac{2}{\theta}$, and $q = s = \frac{2}{2-(\theta+\epsilon)}$, $0 < \theta < 2$ and $\epsilon > 0$.

In addition, we shall require that $p = q$. Therefore $\theta = \frac{2-\epsilon}{2}$; which implies that $p = q = r = s = \frac{4}{2-\epsilon}$,$(\epsilon > 0)$.

By selecting $C'_{\frac{4}{2-\epsilon}} = kc_0c_pc_q = kc_0c_{\frac{4}{2-\epsilon}}c_{\frac{4}{2-\epsilon}}$ and $C''_{\frac{4}{2-\epsilon}} = bc_0c_pc_q = bc_0c_{\frac{4}{2-\epsilon}}c_{\frac{4}{2-\epsilon}}$; and since

$\|K*g\|_{\frac{4}{2-\epsilon},\frac{4}{2-\epsilon}} \leq C_{\frac{8}{3(2-\epsilon)}}\ \|g\|_{\frac{8}{3(2-\epsilon)}}$ follows directly from the following imbedding lemma (namely Lemma 2.2) for the initial data; this in turns will conclude the proof of our Lemma 2.1.

Lemma 2.2 *Let $F(x,t) = K*g$. Assume that $g \in L^r(R^2)$; $1 < r < \infty$.*

If for $p > 1$ and $q > 1$ with $\frac{1}{p} + \frac{1}{2q} = \frac{1}{r}$, then $F(x,t) \in L^{p,q}(R^2xR^+)$ and $\|F\|_{p,q} \leq c(r)\|g\|_r$; where $c(r)$ is some constant depending on r and the dimension.

Proof:
Using (15) we have $|F(x,t)| \leq c_0 \int_{R^2} \frac{|g(x-y)|dy}{\left(|y|+t^{1/2}\right)^2}$

By taking the $L^q(R+)$ norm of both sides, we obtain

$$\left(\int_{R+} |F(x,t)^q dt|\right)^{1/q} \leq c_0\left[\int_{R+}\left\{\int_{R^2}\frac{|g(x-y)|dy}{\left(|y|+t^{1/2}\right)^2}\right\}^q dt\right]^{1/q} \tag{25}$$

Applying the integral inequality of Minkowski on the right hand side to get

$$|F(x,.)|_q \leq c_0\int_{R^2}|g(x-y)|\left[\int_{R+}\frac{dt}{\left(|y|+t^{1/2}\right)^{2q}}\right]^{1/q} \tag{26}$$

from which we obtain

$$|F(x,.)|_q \leq c_0 d\int_{R^2}\frac{|g(y)|dy}{|x-y|^{2-\frac{1}{q}}} \tag{27}$$

for some constant d that does not exceed $\int_0^\infty \frac{dt}{\left(1+t^{1/2}\right)^{2q}}$.

Since $g \in L^r(R^2)$ and $1 < p < \infty$, we get for $\frac{1}{p} = \frac{1}{r} - \frac{1}{2q}$ the necessary conditions to apply the Benedek-Panzone theorem to reach $\|F\|_{p,q} \leq c\,(r)\,\|g\|_r$; where $c(r) = C(c_0, d, r)$.

This concludes the proof of Lemma 2.2.

3 Main Results

These are the main results of the paper.

Non-local or long range diffusion in the $L^{p,q}$ norms

Let us now consider a modification to our model in $(1)^*$ and $(2)^*$; namely:

$$u_t - \Delta^{(2)}u = a_1u^4 + a_2uv^3 + a_3u^2v^2 + a_4u^3v + a_5v^4 + a_6\Delta u^\alpha \tag{28}$$

and

$$v_t - \Delta^{(2)}v = -b_1v^4 - b_2uv^3 - b_3u^2v^2 - b_4u^3v - b_5u^4 + b_6\Delta v^\alpha; \tag{29}$$

where u^4 represents interaction between 4 species in the same population, uv^3 represents the interaction between one species from the first population and 3 species from the second population, and so on.

We will add these two initial conditions to equations (28) and (29) respectively as follows:

$$u(x,0) = f(x) \tag{30}$$

and

$$v(x,0) = k(x) \tag{31}$$

Here $u = u(x,t),\;\; v = v(x,t)\;\;, x \in R^2\;, and\; t \in R^+$.

We shall now convert equation (28) together with the initial condition (30) to an integral equation , then show the existence and uniqueness of long range diffusive solutions to it. Of course, the same procedure can be employed to equation (29) together with the initial condition (31)

To do so, let us define a new Kernel $K_2(x,t) = t^{-\frac{1}{2}}\phi(xt^{-\frac{1}{4}})$; where $K_2 \in C^\infty(R^2)$ is the fundamental solution to the homogeneous PDE:

$$u_t - \Delta^{(2)}u = 0.$$

Also we may use $K_2(x,t) = \Delta K$ and then use (15).

At any rate, K_2 can be approximated as follows:

$$0 \leq |K_2(x,t)| \leq \frac{c_1}{\left(|x| + t^{1/4}\right)^4}; \tag{32}$$

where c_1 is some positive constant.

Our goal now is to calculate α that makes the solution to our equations (28), (29), (30), and (31) exist and unique.

To do so, let us covert equation (28) together with the initial condition (30) into an integral equation and do the same thing for equations (29) and (31) to obtain:

$$u = K_2 \otimes [a_1u^4 + a_2uv^3 + a_3u^2v^2 + a_4u^3v + a_5v^4 + a_6\Delta u^\alpha] + K * f \tag{33}$$

and

$$v = K_2 \otimes [-b_1v^4 - b_2uv^3 - b_3u^2v^2 - b_4u^3v - b_5u^4 + b_6\Delta v^\alpha] + K * k \tag{34}$$

First of all, we shall proof the following lemma for the initial data in equation (33) and the proof for equation (34) is analogous.

Lemma 3.1 *If $f \in L^q(R^2)$ and $0 \leq |K(x,t)| \leq \frac{c_1}{\left(|x|+t^{1/2}\right)^2}$; where c_1 is a constant and $t \succ 0$ then $K * f \in L^{3q}$.*

Proof:
$K * f \leq c_1 \int_{R^2} \frac{f(y)dy}{\left(|x-y|+t^{1/2}\right)^2}$
Let us take the p norm in t to both sides of the above inequality; namely

$$\|K * f\|_p \leq \left\| c_1 \int_{R^2} \frac{f(y)dy}{\left(|x-y|+t^{1/2}\right)^2} \right\|_p$$

Apply the Minkowski's integral inequality on the right hand side to obtain:

$$\begin{aligned}\|K * f\|_p &\leq c_1 \int_{R^2} |f(y)| \left[\int_{R^+} \frac{dt}{\left(|x-y|+t^{1/2}\right)^{2p}}\right]^{\frac{1}{p}} dy \\ &\leq c_1\beta \int_{R^2} |f(y)| \left[\int_{R^+} \frac{1}{\left(|x-y|+t^{1/2}\right)^{2p-4}}\right]^{\frac{1}{p}} dy \\ &= c_1\beta \int_{R^2} \frac{|f(y)|dy}{\left(|x-y|+t^{1/2}\right)^{2-\frac{4}{p}}} \text{ ; where } \beta \text{ is a constant.}\end{aligned}$$

Let us now take the q norm in x of the above inequality to obtain:

$$\|K*f\|_q \leq c_1\beta \left\| \int_{R^2} \frac{|f(y)|dy}{\left(|x-y|+t^{1/2}\right)^{2-\frac{4}{p}}} \right\|_q$$

Using the Benedek-Panzone Theorem, we find out that the right hand side of the above inequality is less than or equal to $C\|f\|_q$ if and only if $\frac{1}{p}=\frac{1}{q}-\frac{2}{p}$; where C is a constant.

This means that $p=3q$ *Q.E.D.*

We will now prove our main Theorem in this paper. But first let $z(x,t)=(u(x,t),v(x,t))$.

Theorem 3.2 *If $z(x,t)\in L^{2,2}(R^2xR^+)$, then the solution $z(x,t)$ to equations (28), (29), (30), and (31) exists and it is unique if $\alpha \succ \frac{3}{2}$.*

Proof: We will use the same process used earlier in proving Lemma 2.1

and other Lemmas; but we will be brief here and without repetition. Namely, by (32) we have:

$$|K_2(x,t)| \leq \frac{c_1}{\left(|x|+t^{1/4}\right)^4} = \frac{c_1}{\left(|x|+t^{1/4}\right)^{2+4-2}}$$

Therefore

$$\frac{1}{q}=\frac{\alpha}{p}-\frac{2}{2+4}$$

i.e., $$1 \prec \frac{p}{\alpha} \prec 3$$

Let $p=q$, then $p=3(\alpha-1)$. But since $1 \prec \frac{p}{\alpha} \prec 3$ we will have:

$$\alpha \prec 3(\alpha-1) \prec 3\alpha$$

Which yields that $\alpha \succ \frac{3}{2}$ as required. But to complete the proof to our Theorem we need to construct a contraction mapping from $L^p(R^2\text{x}R^+)$ into $L^p(R^2\text{x}R^+)$.To do so, we must equate all the exponents of p we got earlier and also in Lemma 3.1; namely

$$3q=3(\alpha-1)$$

Thus,

$$q=\alpha-1$$

Hence, the contraction mapping becomes:

$$\|T(z)\|_{3(\alpha-1)} \leq C(\alpha)\|z\|^{\alpha}_{3(\alpha-1)} + \|l\|_{3(\alpha-1)}; \tag{35}$$

where $l=(f,k)$. We shall now compare (35) with the mapping

$$y = Ax^{\alpha} + B, \tag{36}$$

where A and B are constants.

As we know, Ax^{α} grows faster than a linear function and it is convex.

For $B = 0$, we have only one non-zero root of (36).

Since the graph of Ax^{α} and $y = x$ will intersect in only one non-zero point.

For the same reason, if $0 \prec B \prec \in$ where $\in$ is sufficiently small, we have two roots.

Now, if we let x_1 to be the smallest root, then $y(x) \leq x$ whenever $0 \preceq x \preceq x_1$.

This implies that $\|T(u)\| \preceq x_1$ if $\|u\| \preceq x_1$.

Now, by using (35), we have

$$T(z) = d_1 K_2 \otimes z^{\alpha} + l \tag{37}$$

where d_1 is a constant.

Therefore, if x_1 is chosen to be small enough, then the mapping $T(z)$ in (37) will be a contraction mapping that maps the ball of radius x_1 onto itself.

Hence, the solution to our equations (28), (29), (30), and (31) for which $z = T(z)$ with $z = (u, v)$ will exist and be unique in the ball of radius x_1 such that x_1 depends on the size of the initial data.

This completes the proof of Theorem 3.2.

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