Convergence Theorems for Common Fixed Point of a Finite Family of Nonself $I_i$-Asymptotically Quasi-Nonexpansive Mappings

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Abstract

In this paper, we consider the weak and strong convergence of implicit iteration process to a common fixed point of a finite family of nonself $I_i$-asymptotically quasi-nonexpansive mappings. Our results of this paper improve and extend the corresponding results of Shazad [8], Temir [5] and Gul [J. Math. Anal. Appl. 329(2007), 759-765].

Mathematics Subject Classification: 47H09, 47J25

Keywords and Phrases: Nonself $I_i$-asymptotically quasi-nonexpansive mapping, common fixed point, Convergence theorems

1 Introduction

Let $C$ be a nonempty subset of a real normed linear space $X$. Let $T$ be a self-mapping of $C$. $T$ is said to be asymptotically nonexpansive if there exists a real sequence $\{\lambda_n\} \subset [0, +\infty)$, with $\lim_{n \to \infty} \lambda_n = 0$, such that $\|T^n x - T^n y\| \leq (1 + \lambda_n)\|x - y\|$, $\forall x, y \in C$.

It was proved in [2] that if $X$ is uniformly convex and if $C$ is bounded closed and convex subset of $X$, then every asymptotically nonexpansive mapping has a fixed point.

$T$ is called $I$-asymptotically quasi-nonexpansive on $C$ if there exists sequence $\{v'_n\} \subset [0, \infty)$ with $\lim_{n \to \infty} v'_n = 0$ such that $\|T^n u - p\| \leq (v'_n + 1)\|T^n u - p\|$ for all $u \in C, p \in F(T) \cap F(I)$ and $n = 1, 2, \ldots$.

Remark 1.1. From above definitions, it is easy to see that if $F(T)$ is nonempty, an asymptotically nonexpansive mapping must be $I$-asymptotically quasi-nonexpansive. But the converse does not hold.

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This work was supported by the Scientific Research Foundation of Kunming University.
In the past few decades, many results on fixed points on asymptotically non-expansive, quasi-nonexpansive and asymptotically quasi-nonexpansive mappings in Banach space and metric spaces (see, e.g., [6,7,9]). Very recently, Rhoades and Temir [5] studied the convergence theorems for $I$-nonexpansive mappings, Temir and Gul [10] studied the convergence theorems for $I$-asymptotically quasi-nonexpansive mapping in Hilbert space. But when we look back with regret on [5,10], them does not give any strong convergence theorems.

In most papers which concern the iteration methods, the Mann iteration scheme has been studied and the mapping $T$ has been assumed to map $C$ into itself. The convexity of $C$ then ensures that the sequence $\{x_n\}$ is well defined. If, however, $C$ is a proper subset of the real Banach space $X$ and $T$ maps $C$ into $X$ (as the case in many applications), then the sequence $\{x_n\}$ may not be well defined. One method that has been used to overcome this in the case of single mapping $T$ is to generalize the iteration scheme by introducing a retraction $P : X \to C$ as follows: for $x_1 \in C$,

$$x_{n+1} = P((1-a_n)x_n + a_nT(PT)^{n-1}x_n), \quad n \geq 1,$$

(1.1)

For nonself nonexpansive mappings, some authors (see, e.g., [9,12]) have studied the strong and weak convergence theorems in Hilbert space or uniformly convex Banach spaces.

As an important generalization of the class of asymptotically nonexpansive self-mappings, Chidume [3] in 2003 generalize nonexpansive, asymptotically nonexpansive, uniformly $L$-Lipschitzian to

**Definition 1.1.** Let $C$ be a nonempty subset of a real normed space $X$. Let $P : X \to C$ be a nonexpansive retraction of $X$ onto $C$.

(1) A nonself mapping $T : C \to X$ is called asymptotically if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that for every $n \in \mathcal{N}$,

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x - y\|, \quad \text{for every } x, y \in C.$$

(2) $T$ is said to be uniformly $L$-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\|, \quad \text{for every } x, y \in C.$$

(3) If let $T, I : C \to X$, the mapping $T$ is said to be $\Gamma$-Lipschitzian if there exists $\Gamma \geq 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq \Gamma\|I(PI)^{n-1}x - I(PI)^{n-1}y\|, \quad \text{for every } x, y \in C.$$


Recently, concerning the convergence problems of an implicit iterative process to a common fixed point for a finite family of asymptotically nonexpansive
mappings have been obtained by a number of authors (see, e.g. [5,13]). Xu and Ori [13], in 2001, introduced an implicit iteration process for a finite family of nonexpansive mappings. They proved the weak convergence of the sequence \( \{x_n\} \) to a common fixed point for a finite family of nonexpansive mappings defined in Hilbert space.

Gu and Lu [3], in 2006, studied the weak and strong convergence of implicit iteration process with errors to a common fixed point for a finite family of nonexpansive mappings in Banach spaces.

**Definition 1.2.** Let \( T_i : X \to C, i \in \{1, ..., N\}, T_i \) is nonself I-asymptotically quasi-nonexpansive mappings, \( I_i \) is nonself asymptotically nonexpansive. Then an iterative scheme is the sequences of mappings \( \{x_n\} \) defined by, for given \( x_1 \in C \),

\[
x_{n+1} = P(a_nT_{i(n)}(PT_{i(n)})^{k(n)-1}x_{n+1} + b_nx_n + c_n u_n) \quad n \geq 1,
\]

where \( \{a_n\}, \{b_n\}, \{c_n\} \) are real sequences in \([\delta, 1 - \delta]\) for some \( \delta \in (0, 1) \) with \( a_n + b_n + c_n = 1 \) and \( n = (k(n) - 1)N + i(n), i(n) \in \{1, ..., N\} \).

Motivated by above works, we consider the iteration scheme (1.2) to approximating common fixed points for a finite family of nonself \( I_i \)-asymptotically quasi-nonexpansive mappings \( T_i \), and obtain the weak and strong convergence theorems for such mappings in uniformly convex Banach spaces.

2 Preliminaries.

Throughout this paper, we denote the set of all fixed points of a mapping \( T \) by \( F(T) \).

For approximating fixed points of nonexpansive mappings, Senter and Dotson [7] introduced a Condition (A). Later on, Maiti and Ghosh [4], Tan and Xu [9] studied the Condition (A) and pointed out that Condition (A) is weaker than the requirement of semi-compactness on mapping. For a finite family mappings, the condition (A) can be written to condition (B) as follow.

**Definition 2.1.** The mappings \( T_i : C \to C, (i \in \{1, ..., N\}) \) are said to satisfy condition (B) if there exists a nondecreasing function \( f : [0, +\infty) \to [0, +\infty) \) with \( f(0) = 0, f(r) > 0 \) for all \( r \in (0, +\infty) \) such that \( \max_{1 \leq i \leq N} \|x - T_i x\| \geq f(d(x, F)) \) for all \( x \in C \), where \( F = \bigcap_{i=1}^{N} F(T_i) \) and \( d(x, F) = \inf \{d(x, x^*) : x^* \in F\} \).

In what follows we shall use the following results.

**Lemma 2.1** [2]. Let \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) and \( \{\mu_n\} \) be four nonnegative real sequences satisfying \( \alpha_{n+1} \leq (1 + \gamma_n)(1 + \mu_n)\alpha_n + \beta_n \) for all \( n \geq 1 \). If \( \sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} \gamma_n < \infty \) and \( \sum_{n=1}^{\infty} \beta_n < \infty \), then \( \lim_{n \to \infty} \alpha_n \) exists.

**Lemma 2.2.** [6] Let \( E \) be a real uniformly convex Banach space and \( 0 \leq p \leq t_n \leq q < 1 \) for all positive integer \( n \geq 1 \). Also suppose \( \{x_n\} \) and
\{y_n\} are two sequences of \(E\) such that \(\limsup_{n \to \infty} \|x_n\| \leq r\), \(\limsup_{n \to \infty} \|y_n\| \leq r\) and \(\limsup_{n \to \infty} \|t_n x_n + (1 - t_n)y_n\| = r\) hold for some \(r \geq 0\), then \(\lim_{n \to \infty} \|x_n - y_n\| = 0\).

**Lemma 2.3.** [1] Let \(X\) be a real uniformly convex Banach space, \(C\) a nonempty closed convex subset of \(X\), and let \(T: C \to X\) be nonself asymptotically nonexpansive mapping with a sequence \(\{k_n\} \subset [1, \infty)\) and \(k_n \to 1\) as \(n \to \infty\). Then \(E - T\) is demiclosed at zero.

**3 Main Results**

**Lemma 3.1.** Let \(X\) be a uniformly convex Banach space, \(C\) be a nonempty bounded and closed convex subset of \(X\), \(\{T_i : i \in \{1, 2, \ldots, N\}\} : C \to X\) be \(N\) \(I_i\)-asymptotically quasi-nonexpansive nonself-mappings with sequences \(\{v_{i_n}\} \subset [0, \infty)\) such that \(\sum_{n=1}^{\infty} u_{i_n} < \infty\) and \(I_i : i \in \{1, \ldots, N\} : C \to X\) be \(N\) asymptotically nonexpansive nonself-mappings with \(\{v_{i_n}\} \subset [0, \infty)\) such that \(\sum_{n=1}^{\infty} v_{i_n} < \infty\) and \(F = \bigcap_{i=1}^{N} (F(T_i) \cap F(I_i)) \neq \emptyset\). Then the implicitly iterative sequence \(\{x_n\}\) is generated by (1.2) converges strongly to a common fixed point in \(F\) if and only if \(\liminf_{n \to \infty} d(x_n, F) = 0\).

**Proof.** Since \(C\) is bounded, there exists \(M > 0\) such that \(\|x_n - u_n\| \leq M\), for all \(n \in N\). For any \(p \in F = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i) \neq \emptyset\).

\[
\|x_{n+1} - p\| = \|b_n x_n + a_n T_{i(n)} (PT_{i(n)})^{k(n)-1} x_{n+1} + c_n u_n - p\|
\leq (1 - a_n) \|x_n - p\| + a_n \|T_{i(n)} (PT_{i(n)})^{k(n)-1} x_{n+1} - p\| + c_n \|u_n - x_n\|
\leq (1 - a_n) \|x_n - p\| + a_n (1 + u_{ik}) \|I_{i(n)} (PI_{i(n)})^{k(n)-1} x_{n+1} - p\| + c_n M
\leq (1 - a_n) \|x_n - p\| + a_n (1 + u_{ik}) (1 + v_{ik}) \|x_{n+1} - p\| + c_n M
\leq (1 - a_n) \|x_n - p\| + (a_n + u_{ik} + v_{ik}) \|x_{n+1} - p\| + c_n M.
\]

Transposing and simplifying above inequality and noticing that \(a_n \in [\delta, 1 - \delta]\). We have

\[
(1 - a_n) \|x_{n+1} - p\| \leq (1 - a_n) \|x_n - p\| + (u_{ik} + v_{ik}) \|x_{n+1} - p\| + c_n M
\leq (1 - a_n) \|x_n - p\| + (u_{ik} + v_{ik}) \left(1 - \frac{a_n}{\delta}\right) \|x_{n+1} - p\| + c_n M \left(1 - \frac{a_n}{\delta}\right).
\]

and

\[
\frac{\delta - u_{ik} - v_{ik}}{\delta} \|x_{n+1} - p\| \leq \|x_n - p\| + c_n \frac{M}{\delta}.
\]

Since \(\Sigma_{k=1}^{\infty} u_{ik} < \infty\) and \(\Sigma_{k=1}^{\infty} v_{ik} < \infty\) for all \(i \in \{1, 2, \ldots, N\}\), thus \(\lim_{k \to \infty} u_{ik} = 0, \lim_{k \to \infty} v_{ik} = 0\), there exists a natural number \(n_0\), as \(k > \frac{n_0}{N} + 1\), i.e. \(n > n_0\) such that \(\delta - u_{ik} - v_{ik} > 0\) and \(u_{ik} + v_{ik} < \frac{\delta}{2}\). Then we have

\[
\|x_{n+1} - p\| \leq (1 + w_{ik}) \|x_n - p\| + c_n M_1.
\]
where \( M_1 = \frac{M}{\delta}, w_{ik} = \frac{u_{ik} + v_{ik}}{\delta - u_{ik} - v_{ik}} < \frac{2}{\delta}(u_{ik} + v_{ik}) < \frac{2}{\delta}u_{ik}, \) therefore \( \sum_{k=1}^{\infty} w_{ik} < \frac{2}{\delta - u_{ik} - v_{ik}} < \infty, \) for all \( i. \)

This implies that \( d(x_{n+1}, F) \leq (1 + w_{ik})d(x_n, F) + c_n M_1. \) From Lemma 2.1 \( \lim_{n \to \infty} d(x_{n+1}, F) \) exists and by the hypothesis \( \liminf_{n \to \infty} d(x_n, F) = 0. \) We have \( \lim_{n \to \infty} d(x_n, F) = 0. \)

Let \( \varepsilon > 0 \) since \( \lim_{n \to \infty} d(x_n, F) = 0, \) there exist natural number \( N_1 \) such that when \( n \geq N_1, d(x_n, F) < \frac{\varepsilon}{3}. \) Thus, there exists \( x^* \in F \) such that for above \( \varepsilon \) there exists positive integer \( N_2 \geq N_1 \) such that as \( n \geq N_2, \| x_n - x^* \| < \frac{\varepsilon}{2}. \)

Now for arbitrary \( n, m \geq N_2, \) consider \( \| x_n - x_m \| \leq \| x_n - x^* \| + \| x_m - x^* \| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon. \) This implies that \( \{ x_n \} \) is a cauchy sequence in \( K, \) therefor it converges to a point, say \( p \in K. \) And \( \lim_{n \to \infty} d(x_n, F) = 0 \) gives that \( d(p, F) = 0. \) By the routine proof, we know \( F \) is closed. Thus \( p \in F. \) The proof is completed.

**Lemma 3.2.** Let \( X, C, \{ x_n \} \) be same as Lemma3.1, \( \{ T_i : i \in \{1, 2, ..., N\} \} : C \to X \) be \( N \) uniformly \( \Gamma \)-Lipschitzian \( I_i \)-asymptotically quasi-nonexpansive nonself-mappings with sequences \( \{ v_{in} \} \subset [0, \infty) \) such that \( \sum_{i=1}^{\infty} v_{in} < \infty \) and \( I_i : i \in \{1, \ldots, N\} : C \to X \) be \( N \) uniformly \( L \)-Lipschitzian asymptotically nonexpansive nonself-mappings with \( \{ u_{in} \} \subset [0, \infty) \) such that \( \sum_{i=1}^{\infty} u_{in} < \infty \) and \( F = \cap_{i=1}^{N} F(T_i) \cap F(I_i) \neq \emptyset. \) If \( F \neq \emptyset \) then \( \lim_{n \to \infty} \| T_i x_n - x_n \| = 0, \forall l = 1, 2, \ldots, N. \)

**Proof.** By Lemma 3.1 for any \( p \in F, \lim_{n \to \infty} \| x_n - p \| \) exists. Let \( \lim_{n \to \infty} \| x_n - p \| = d. \) Since

\[
\| T_i(n)(PT_i(n))^{k(n)-1}x_{n+1} - p + c_n(u_n - x_n) \|
\leq \Gamma l \| I_i(n)(T_i(n)(PT_i(n))^{k(n)-2}x_{n+1}) - p \| + c_n M
\leq \Gamma L \| T_i(n)(PT_i(n))^{k(n)-2}x_{n+1} - p \| + c_n M \leq \cdots \leq (\Gamma L)^n \| x_{n+1} - p \| + c_n M.
\]

We have \( \limsup_{n \to \infty} \| T_i(n)(PT_i(n))^{k(n)-1}x_{n+1} - p + c_n(u_n - x_n) \| \leq d. \)
\( \| x_n - p + c_n(u_n - x_n) \| \leq \| x_n - p \| + c_n M \) which implies \( \limsup_{n \to \infty} \| x_n - p + c_n(u_n - x_n) \| \leq d. \)

And \( \lim_{n \to \infty} \| x_{n+1} - p \| = d \) means that \( \lim_{n \to \infty} \| a_n[T_i(n)(PT_i(n))^{k(n)-1}x_{n+1} - p + c_n(u_n - x_n)] + (1 - a_n)[x_n - p + c_n(u_n - x_n)] \| = d. \) By Lemma 2.2, we have

\[
\lim_{n \to \infty} \| T_i(n)(PT_i(n))^{k(n)-1}x_{n+1} - x_n \| = 0
\]

Hence \( \| x_{n+1} - x_n \| = a_n \| T_i(n)(PT_i(n))^{k(n)-1}x_{n+1} - x_n \| + c_n \| u_n - x_n \| \to 0. \)

From \( \| T_i(n)(PT_i(n))^{k(n)-1}x_n - x_n \| \leq \| x_{n-1} - T_i(n)(PT_i(n))^{k(n)-1}x_n \| + \| x_n - x_n \|, \)
it follows that

\[
\lim_{n \to \infty} \| T_i(n)(PT_i(n))^{k(n)-1}x_n - x_n \| = 0 \quad (3.3)
\]

Notice that for each \( n > N, n = (n - N)(mod \ N) \) and \( n = (k(n) - 1)N + i(n), \) hence \( n - N = ((k(n) - 1) - 1)N + i(n) = (k(n - N) - 1)N + i(n - N), \) that is \( k(n - N) = k(n) - 1 \) and \( i(n - N) = i(n). \)
From (3.3)
\[ \|x_n - T_nx_n\| \leq \|x_n - T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n\| + \|T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n - T_nx_n\| \]
\[ \leq \|x_n - T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n\| + L\|T_{i(n)}(PT_{i(n)})^{k(n)-2}x_n - x_n\| \]
\[ \leq \|x_n - T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n\| + \Gamma L\|T_{i(n)}(PT_{i(n)})^{k(n)-2}x_n - T_{i(n)-N}(PT_{i(n)-N})^{k(n)-2}x_n - x_n\| + \|x_{n-N} - x_n\| \]
\[ \leq \|x_n - T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n\| + \Gamma L(1+\Gamma L)\|x_n - x_{n-N}\| + \Gamma L\|T_{i(n)-N}(PT_{i(n)-N})^{k(n)-1}x_n - x_n\| \rightarrow 0 \quad (n \to \infty). \]

This implies that \( \lim_{n \to \infty} \|T_nx_n - x_n\| = 0 \). Now for all \( l = \{1, 2, ..., N\} \)
\[ \|x_n - T_{n+l}x_n\| \leq \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l}x_{n+l}\| + \|T_{n+l}x_{n+l} - T_{n+l}x_n\| \]
\[ \leq \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l}x_{n+l}\| + \Gamma L\|x_{n+l} - x_n\| \]
\[ \leq \|x_{n+l} - T_{n+l}x_{n+l}\| + (1+\Gamma L)\|x_n - x_{n+l}\| \to 0. \]

So \( \lim_{n \to \infty} \|T_{n+l}x_n - x_n\| = 0 \) for all \( l = \{1, 2, ..., N\} \). Consequently, we have
\[ \lim_{n \to \infty} \|T_{i}x_n - x_n\| = 0. \quad (3.4) \]

The proof is completed.

**Theorem 3.3.** Let \( X \) be a uniformly convex Banach space and \( C, T_i, I_i, \{x_n\} \) be same as in Lemma 3.2. If \( F \neq \emptyset \), then \( \{x_n\} \) converges weakly to a common fixed point of \( \{T_i, i \in \{1, ..., N\}\} \).

**Proof.** Let \( p \in F \). Then, as Lemma 3.2, it follows \( \lim_{n \to \infty} \|x_n - p\| \) exists and so for \( n \geq 1 \), \( \{x_n\} \) is bounded on \( C \). Since \( X \) is uniformly convex, every bounded subset of \( X \) is weakly compact the boundedness of \( \{x_n\} \) in \( C \), there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( \lim_{n \to \infty} x_{n_k} = p \) weakly. We assume that \( n_k = i(modN) \), where \( i \) is some positive integer in \( \{1, 2, ..., N\} \). Otherwise, we can take a subsequence \( \{x_{n_k}\} \subset x_{n_k} \) such that \( n_k = i(modN) \).

For \( l \in \{1, 2, ..., N\} \), there exists an integer \( j \in \{1, 2, ..., N\} \) such that \( n_{k+j} = l(modN) \). For all \( l \in \{1, 2, ..., N\} \), from (3.4) we have \( \lim_{n \to \infty} \|T_{i}x_n - x_n\| = 0 \). Also, by Lemma 2.3, for each \( l \in \{1, 2, ..., N\} \) we know that \( p \in F(T_l) \). By arbitrariness of \( l \in \{1, 2, ..., N\} \), we have \( p \in F = \bigcap_{l=1}^{N} F(T_l) \). If \( F \) is a singleton, then the proof is complete. For \( p, q \in F \), we assume that \( F \) is not singleton. Suppose \( p, q \in \omega(\{x_n\}) \), where \( \omega(\{x_n\}) \) denotes the weak limit set of \( \{x_n\} \). Let \( x_{n_k} \) and \( x_{m_l} \) be two subsequences of \( \{x_n\} \) which converge weakly to \( p \) and \( q \), respectively. By Lemma 3.2 and 2.3, we have \( p \in \bigcap_{l=1}^{N} F(T_l) \) and \( q \in \bigcap_{l=1}^{N} F(T_l) \). Therefore \( T_lp = p \) for all \( l \in \{1, 2, ..., N\} \) and \( T_lq = q \) for all \( l \in \{1, 2, ..., N\} \).
For the uniqueness, assume that \( p \neq q \) and \( \{x_n\} \to p, \{x_m\} \to q \). By Opial’s condition, we get that

\[
\lim_{n \to \infty} \|x_n - p\| = \lim_{k \to \infty} \|x_{nk} - p\| \leq \lim_{k \to \infty} \|x_{nk} - q\| = \lim_{n \to \infty} \|x_n - p\| \leq \lim_{j \to \infty} \|x_{mj} - q\| \leq \lim_{j \to \infty} \|x_{mj} - p\| \leq \lim_{n \to \infty} \|x_n - q\|.
\]

This is a contradiction. Thus \( \{x_n\} \) converges weakly to a common fixed point of \( \{T_i, i \in \{1, \ldots, N\}\} \). The proof is completed.

**Theorem 3.4.** Let \( X \) be a uniformly convex Banach space and \( C, T_i, I_i, \{x_n\} \) be same as in Lemma 3.2. If \( F \neq \emptyset \) and \( I_i, T_i \) satisfy the Condition (B), then \( \{x_n\} \) converges strongly to a common fixed point of \( \{T_i, i \in \{1, \ldots, N\}\} \).

**Proof.** By Lemma 3.1, for all \( p \in F \), from (3.2), \( \|x_{n+1} - p\| \leq (1 + w_{ik})\|x_n - p\| + c_n M \) for \( n \geq 1 \) with \( \sum_{k=1}^{\infty} w_{ik} < \infty \) and \( \sum_{n=1}^{\infty} c_n < \infty \), for all \( i \). This implies that \( d(x_{n+1}, F) \leq (1 + w_{ik})d(x_n, F) + c_n M \). By Lemma 2.1 \( \lim_{n \to \infty} d(x_{n+1}, F) \) exists.

From Lemma 3.2, \( \lim_{n \to \infty} \|T_l x_n - x_n\| = 0, \forall l = 1, 2, \ldots, N \). The condition (B) guarantees that \( \lim_{n \to \infty} f(d(x_n, F)) = 0 \). Since \( f \) is a nondecreasing function and \( f(0) = 0 \), it follows that \( \lim_{n \to \infty} d(x_n, F) = 0 \). By Lemma 3.1, we know that \( \{x_n\} \) is a cauchy sequence. Thus, \( \{x_n\} \) converges strongly to a common fixed point of \( \{T_i, i \in \{1, \ldots, N\}\} \).

**References**


Received: January, 2009