A Note on Hilbert-Schmidt Hankel-Operators between Differently Weighted Spaces

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Abstract
The paper we investigates Hankel operators $H_f : F^2_1 \rightarrow L^2_2$ with anti-holomorphic symbols $f = \sum_{k=0}^{\infty} b_k \overline{z}^k \in L^2_1$, where $F^2_1$ is a generalized Fock space and $L^2_{2n}$ is the $L^2$ space with weight $e^{-|z|^m}$. We show that each Hankel operator with anti-holomorphic $L^2_{2n}$-symbol is Hilbert-Schmidt. This result complements previous research of [5]. There it is shown that in the general case of Hankel operators $H_f : F^2_n \rightarrow L^2_m$ and $n < m$ each Hankel operator with polynomial symbol is Hilbert-Schmidt.

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1 Introduction

We define the generalized Fock space $F^2_n$ by

$$F^2_n := F^2(\mathbb{C}, |z|^n) := \left\{ g : \mathbb{C} \rightarrow \mathbb{C} \mid g \text{ entire and } \|g\|_n^2 < \infty \right\},$$

where

$$\|g\|_n^2 := \int_{\mathbb{C}} |g(z)|^2 e^{-|z|^n} d\lambda(z),$$
where \( d\lambda \) denotes the Lebesgue measure in \( C \cong \mathbb{R}^2 \). Additionally, let
\[
L_n^2 := L^2(C, |z|^n) := \{ g : C \to C \mid g \text{ measurable and } \|g\|_n^2 < \infty \} / \ker \|n\).
\]

It is standard to see that \( \mathcal{F}_n^2 \) is a closed subspace of \( L_n^2 \) and therefore a Hilbert space.

The Hankel operator, \( H_n^2 : \mathcal{F}_n^2 \to \mathcal{F}_n^{2 \perp} \), with symbol \( f : C \to C \) (\( f \in L_n^2 \)) is given by
\[
H_n^2(h) = (\text{Id} - P_n)(\overline{f}h),
\]
where \( \mathcal{F}_n^{2 \perp} \) denotes the orthogonal space of the generalized Fock space and \( P_n : L_n^2 \to \mathcal{F}_n^2 \) denotes the Bergman projection. We ignore the dependence of \( H_n^2 \) on \( n \) in the following if the choice of \( n \) is obvious. Note, that the above definition surely makes sense for all \( h \) that satisfy \( \overline{f}h \in L_n^2 \). In general, the Hankel operator will not be globally defined. Let \( \{u_{k,n} := \frac{z^k}{c_{k,n}} \mid k \in \mathbb{N} \} \) be the natural basis of \( \mathcal{F}_n^2 \) and \( c_{k,n} \) the moments corresponding to \( n \). That is
\[
c_{k,n}^2 = \langle z^k, z^k \rangle_n = \int_C |z^k|^2 e^{-|z|^n} d\lambda(z) = \|z^k\|_n^2, \ k, n \in \mathbb{N}.
\]

In [5] we have proved the following result:

**Theorem 1.1.** Let \( \overline{f} \) be a anti-holomorphic symbol and consider the Hankel operator \( H_{\overline{f}} \) as an operator from \( \mathcal{F}_n^2 \) to \( L_n^2 \). Then we have the following result:

(1) For \( n > m \) there are no nontrivial Hilbert-Schmidt Hankel operators with anti holomorphic symbols. That is, if a Hankel operator with symbol \( \overline{f} \) is Hilbert-Schmidt, the symbol \( \overline{f} \) must be constant. Moreover, there are no nontrivial bounded Hankel operators with anti-holomorphic symbols.

(2) For \( n = m \) there are no nontrivial Hilbert-Schmidt Hankel operators with anti holomorphic symbols. However, there are bounded Hankel operators. In particular, the Hankel operator \( H_{\overline{f}} \) is bounded if and only if \( \overline{f} \) is a polynomial of degree \( N \) satisfying \( 2N \leq m \).

(3) For \( n < m \) each Hankel operator with monomial anti-holomorphic symbol is Hilbert-Schmidt and therefore each Hankel operator with polynomial anti-holomorphic symbol is Hilbert-Schmidt. Especially, each Hankel operator with polynomial anti-holomorphic symbol is bounded.

This work complements some earlier work (See [2], [3] and [1]). In this paper we want to prove the following result:
Theorem 1.2. Let $\mathcal{F}$ be an anti-holomorphic symbol and consider the Hankel operator $H_\mathcal{F}$ as an operator from $\mathcal{F}_1^2$ to $L_2^2$. Then all Hankel operators with anti-holomorphic $L_2^2$-symbols are Hilbert-Schmidt.

2 Proof of the main result

In this section we prove Theorem 1.2.

Proof. Let $\mathcal{F} = \sum_{l=0}^{\infty} b_l z^l \in L_1^2$, where $b_l \in \mathbb{C}$. The condition that $\mathcal{F} \in L_1^2$ can be equivalently expressed by

$$\|\mathcal{F}\|_1^2 = \sum_{l=0}^{\infty} |b_l|^2 c_{l,1}^2 < \infty. \quad (1)$$

Now, consider the expression

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} |b_l|^2 c_{k+l,2} \frac{c_{k,1}^2}{c_{k,1}^2} + \sum_{k=0}^{\infty} \sum_{l=0}^{k} |b_l|^2 \left( \frac{c_{k,2}^2 c_{k-l,2}}{c_{k,1}^2} - 2 \frac{c_{k,2}^2}{c_{k-l,1}^2} \right) \geq 0.$$ 

It is obvious, from what we know so far, that if the above sum is finite, the Hankel operator with symbol $\mathcal{F} = \sum_{l=0}^{\infty} b_l z^l$ is Hilbert-Schmidt. Therefore, the aim of the following part of the proof is to show that the above sum is finite. Using the identity

$$c_{k,m}^2 = \frac{2\pi}{m} \Gamma \left( \frac{2k+2}{m} \right),$$

we obtain

$$\frac{c_{k,2}^2 c_{k-l,2}}{c_{k,1}^2} - 2 \frac{c_{k,2}^2}{c_{k-l,1}^2} = \frac{\Gamma(k+1)\Gamma(k-l+1)}{4\Gamma(2k+1)^2} - \frac{\Gamma(k+1)}{\Gamma(2k-2l+2)} \leq \frac{\Gamma(k+1)}{4\Gamma(2k-2l+1)} \left( \frac{\Gamma(k-l+1)}{4\Gamma(2k+1)} - 1 \right) < 0.$$ 

Hence, it suffices to show that the expression

$$\sum_{l=0}^{\infty} \sum_{k=0}^{\infty} |b_l|^2 \frac{c_{k+l,2}}{c_{k,1}^2}$$

is finite. The above sum is equal to

$$\frac{1}{2} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} |b_l|^2 \frac{\Gamma \left( \frac{2(k+l+2)}{2} \right)}{\Gamma \left( \frac{2k+2}{2} \right)} = \frac{1}{2} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} |b_l|^2 \frac{(k+l)!}{(2k+1)!}.$$
According to equation (1) it is sufficient to show that there is a constant $C$ such that for each $l$

$$\sum_{k=0}^{\infty} \frac{c_{k+l,2}^2}{c_{k,1}^2 c_{l,1}^2} < C,$$

i.e.,

$$\sum_{k=0}^{\infty} \frac{\Gamma \left( \frac{2(k+l)+2}{m} \right)}{\Gamma \left( \frac{2k+2}{n} \right) \Gamma \left( \frac{2l+2}{n} \right)} = \sum_{k=0}^{\infty} \frac{(k+l)!}{(2k+1)!(2l+1)!} \leq C.$$

For $k \leq l + 1$ we have

$$\frac{(k+l)!}{(2k+1)!(2l+1)!} \leq \frac{1}{(2k+1)!},$$

and for $k > l + 1$ we have

$$\frac{(k+l)!}{(2k+1)!(2l+1)!} \leq \frac{(2k-1)!}{(2k+1)!(2l+1)!} \leq \frac{1}{(2k+1)(2k)(2l+1)!}.$$

Therefore,

$$\sum_{k=0}^{\infty} \frac{(k+l)!}{(2k+1)!(2l+1)!} \leq \sum_{k=0}^{l+1} \frac{(k+l)!}{(2k+1)!(2l+1)!} + \sum_{k=l+2}^{\infty} \frac{(k+l)!}{(2k+1)!(2l+1)!} \leq \sum_{k=0}^{l+1} \frac{1}{(2k+1)!} + \sum_{k=l+2}^{\infty} \frac{1}{(2k+1)(2k)(2l+1)!} \leq \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k)(2l+1)!} \leq \left(1 + \frac{1}{(2l+1)!}\right) \frac{\pi^2}{6} \leq \frac{\pi^2}{3} = C < \infty$$

for all $l \in \mathbb{N}$. This finishes the proof.
References


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