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Proof of Inequality of Rank of Matrix on Skew Field by Constructing Block Matrix

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Abstract. Using the method of the block matrix construction and generalized elementary transformation given in the paper, readers can easily prove the inequalities of rank of matrix on skew field.

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1. Introduction

Matrix theory on skew field is one of the basic direction in non-exchange algebra research.And the rank of matrix on skew field is an important digital feature of matrix. The inequality (equality)of rank of matrix is one of the important issues newly discussed in the matrix theory. For the proof of matrix-rank inequality(equality),the paper firstly makes use of some matrices to construct block matrix ,then proves the inequality(equality)of rank of matrix by doing generalized elementary transformation to the block matrix above ,therefore some well-known inequalities can be proved by this method.

2. Preparation Knowledge

Let K be a skew field ,$K^{s\times n}$ represent the group of the unit $s \times n$ matrix,I_n represent the identity matrix on the skew field ,$r(A)$ mean the rank of matrix A.

Lemma 1 *Let $A \in K^{s\times n}$,$B \in K^{n\times m}$, and if $AB = 0$, we can get*

$$r(A) + r(B) \leq n.$$

Proof *Suppose $B = (B_1, B_2, \dots , B_m)$, from the known condition $AB = 0$,we can get*

$$A(B_1, B_2, \dots , B_m) = 0,$$

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that is

$$AB_j = 0 (j = 1, 2, \dots, m).$$

And it also means these $B_j (j = 1, 2, \dots, m)$ *are the solutions of right system of homogeneous linear equations* $AX = 0$ *,while we have known that the quantity of the solutions in the solution-based system of* $AX = 0$ *is* $n - r(A)$*, Therefore we have* $r(B) \le n - r(A)$*, namely*

$$r(A) + r(B) \le n.$$

Lemma 2 *Let* $A \in K^{s\times n}$ *,*$B \in K^{n\times m}$*,then*

$$r(AB) \le min(r(A), r(B)).$$

Lemma 3 *Let* $A \in K^{s\times n}$ *,*$B \in K^{n\times m}$*,then*

$$r\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = r(A) + r(B).$$

Generally

$$r\begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_s \end{pmatrix} = r(A_1) + r(A_2) + \dots + r(A_s).$$

Lemma 4 *Assume that* $A \in K^{s\times n}$*,*$B \in K^{n\times m}$ *we can obtain*

$$r(A) + r(B) \le r\begin{pmatrix} A & \star \\ 0 & B \end{pmatrix} \le min(r(A) + m, r(B) + s).$$

Generally suppose that $A \in K^{n_i\times m_i}$*,*$(i = 1, 2, \dots, s; s \ge 2)$ *we gain*

$$r(A_1)+r(A_2)+\dots+r(A_s) \le r\begin{pmatrix} A_1 & *_{12} & \dots & *_{1s} \\ 0 & A_2 & \dots & *_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_s \end{pmatrix} \le min(r(A_i)+\sum_{j=1, j\neq i}^{s} m_j),$$

$$(where \quad i = 1, 2, \dots, s)$$

For form as the below inequality of matrix

$$\sum_{i=1}^{s} r(A_i) \ge \sum_{i=1}^{s} r(B_i),$$

we construct block matrix

$$M = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_s \end{pmatrix},$$

then carry out generalized elementary transformation to matrix M and change it into

$$\begin{pmatrix} B_1 & *_{12} & \dots & *_{1s} \\ 0 & B_2 & \dots & *_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_s \end{pmatrix},$$

so

$$\sum_{i=1}^{s} r(A_i) = r(M) = r \begin{pmatrix} B_1 & *_{12} & \dots & *_{1s} \\ 0 & B_2 & \dots & *_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_s \end{pmatrix} \geq \sum_{i=1}^{s} r(B_i)$$

exists.

3. Proof of inequalities of matrix of rank on skew field

Example 1 *Assume $A \in K^{s\times n}$,$B \in K^{n\times m}$, to prove the Sylvester inequality*

$$r(AB) \geq r(A)+r(B)-n. \tag{1}$$

Proof *The first equation is equivalent to*

$$r(AB) + n \geq r(A) + r(B),$$

that is

$$r(AB) + n = r \begin{pmatrix} I_n & 0 \\ 0 & AB \end{pmatrix}.$$

We construct block matrix

$$M = r \begin{pmatrix} I_n & 0 \\ 0 & AB \end{pmatrix},$$

then carry out generalized elementary transformation to matrix M as follows:

$$M = \begin{pmatrix} I_n & 0 \\ 0 & AB \end{pmatrix} \to \begin{pmatrix} I_n & 0 \\ A & AB \end{pmatrix} \to \begin{pmatrix} I_n & -B \\ A & 0 \end{pmatrix} \to \begin{pmatrix} B & I_n \\ 0 & A \end{pmatrix},$$

namely

$$r(M) = r(AB) + n = r \begin{pmatrix} B & I_n \\ 0 & A \end{pmatrix} \geq r(A) + r(B),$$

thus the first equation comes into existence.

Example 2 *Assume that A, B, C are multiplying-allowed matrices on skew field K,to prove Frobeniua inequality*

$$r(ABC) \geq r(AB)+r(BC)-r(B). \tag{2}$$

Proof *The second equation is equivalent to*

$$r(ABC) + r(B) \geq r(AB) + r(BC).$$

We construct block matrix

$$M = \begin{pmatrix} B & 0 \\ 0 & ABC \end{pmatrix},$$

then carry out generalized elementary transformation to matrixM as follows:

$$M = \begin{pmatrix} B & 0 \\ 0 & ABC \end{pmatrix} \to \begin{pmatrix} B & 0 \\ AB & ABC \end{pmatrix} \to \begin{pmatrix} B & -BC \\ AB & 0 \end{pmatrix} \to \begin{pmatrix} BC & B \\ 0 & AB \end{pmatrix},$$

namely

$$r(B) + r(ABC) = r\begin{pmatrix} BC & B \\ 0 & AB \end{pmatrix} \geq r(AB) + r(BC),$$

therefore the second equation can be established.

Example 3 *Provided that the unit* $n \times n$ *matrix* A *satisfies* $A^2 = A$*(where* A *is called an idempotent matrix),we can get*

$$A^2 = A \Leftrightarrow r(A)+r(I_n-A) = n. \tag{3}$$

Proof *The unit* $n \times n$ *matrix* A *is an idenpotent matrix,then*

$$A^2 = A \Leftrightarrow A^2 - A = 0 \Leftrightarrow r(A_2 - A) = 0.$$

We construct block matrix

$$M = \begin{pmatrix} A & 0 \\ 0 & I_n - A \end{pmatrix},$$

then carry out generalized elementary transformation to matrix M *as follows:*

$$M = \begin{pmatrix} A & 0 \\ 0 & I_n - A \end{pmatrix} \to \begin{pmatrix} A & 0 \\ A & I_n - A \end{pmatrix} \to \begin{pmatrix} A & A \\ A & I_n \end{pmatrix}$$

$$\to \begin{pmatrix} A - A^2 & 0 \\ A & I_n \end{pmatrix} \to \begin{pmatrix} A - A^2 & 0 \\ 0 & I_n \end{pmatrix},$$

namely

$$r(A) + r(I_n - A) = r(A - A^2) + n,$$

so the third equation can come into existence.

Example 4 *Let* $A \in K^{m\times n}$*,then prove*

$$r(I_m-AA')-r(I_n-A'A) = m-n. \tag{4}$$

Proof *The fourth equation is equivalent to*

$$r(I_m - AA') + n = r(I_n - A'A) + m.$$

So we construct block matrix

$$M = \begin{pmatrix} I_m - AA' & 0 \\ 0 & I_n \end{pmatrix},$$

then carry out generalized elementary transformation to matrix M as follows :

$$M = \begin{pmatrix} I_m - AA' & 0 \\ 0 & I_n \end{pmatrix} \to \begin{pmatrix} I_m - AA' & A \\ 0 & I_n \end{pmatrix} \to \begin{pmatrix} I_m & A \\ A' & I_n \end{pmatrix}$$
$$\to \begin{pmatrix} I_m & 0 \\ A' & I_n - A'A \end{pmatrix} \to \begin{pmatrix} I_m & 0 \\ 0 & I_n - A'A \end{pmatrix},$$

namely

$$r(I_m - AA') + n = r(I_n - A'A) + m,$$

so the fourth equation can be proved.

For some inequalities of rank of matrix, we can also make generalized elementary transformation to matrix M firstly, then transform it into a new matrix G ,and then construct block matrix N to transit.At last we carry out generalized elementary transformation to NG and change it into the following form

$$NG \to \begin{pmatrix} B_1 & *_{12} & \dots & *_{1s} \\ 0 & B_2 & \dots & *_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_s \end{pmatrix},$$

so we get

$$\sum_{i=1}^{s} r(A_i) = r(M) = r(G) \geq r(NG) = r \begin{pmatrix} B_1 & *_{12} & \dots & *_{1s} \\ 0 & B_2 & \dots & *_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_s \end{pmatrix} \geq \sum_{i=1}^{s} r(B_i).$$

Example 5 *Proved that A,B are two matrices on the center of skew field K,and $AB = BA$, we can get*

$$r(AB) \leq r(A)+r(B)-r(A+B). \tag{5}$$

Proof *The fifth equation is equivalent to*

$$r(A) + r(B) \geq r(AB) + r(A + B).$$

We construct block matrix

$$M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

then carry out generalized elementary transformation to matrix M as follows :

$$M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \to \begin{pmatrix} A & B \\ 0 & B \end{pmatrix} \to \begin{pmatrix} A + B & B \\ B & B \end{pmatrix} = G,$$

and then select

$$N = \begin{pmatrix} I_n & 0 \\ B & -(A + B) \end{pmatrix},$$

from the known condition $AB = BA$,*we make* N *multiply* G *to get*

$$NG = \begin{pmatrix} I_n & 0 \\ B & -(A+B) \end{pmatrix} \begin{pmatrix} A+B & B \\ B & B \end{pmatrix} = \begin{pmatrix} A+B & B \\ 0 & -AB \end{pmatrix}.$$

Through the work done above ,we can get

$$r(A)+r(B) = r(M) = r(G) = r\begin{pmatrix} A+B & B \\ B & B \end{pmatrix} \geq r\begin{pmatrix} A+B & B \\ 0 & -AB \end{pmatrix} \geq r(AB)+r(A+B),$$

so the fifth equation brings into being.

In addition ,we can also combine the method of structuring block matrix with the basic property of rank of matrix which we have got above to prove some inequalities of rank of matrix.

Example 6 *Let* $A, B, C \in K^{n\times n}$ *,and* $r(C) = n, A(BA + C) = 0$ *,then*

$$r(BA + C) = n - r(A). \tag{6}$$

Proof *With* $A(BA + C) = 0$ *and the* **Lemma 1**,*we can get*

$$r(BA + C) + r(A) \leq n.$$

And with $r(C) = n$ *and*

$$\begin{pmatrix} BA+C & 0 \\ 0 & A \end{pmatrix} \to \begin{pmatrix} BA+C & BA \\ 0 & A \end{pmatrix} \to \begin{pmatrix} C & BA \\ -A & A \end{pmatrix},$$

we easily acquire the below conclusion

$$r(BC + C) + r(A) = r\begin{pmatrix} BA+C & 0 \\ 0 & A \end{pmatrix} = r\begin{pmatrix} C & BA \\ -A & A \end{pmatrix} \geq r(C) = n.$$

consequently, the sixth equation comes to existence.

References

[1] *Songgui Wang ect, Inequality of matrix[M](the second press), Science press, Beijing(2006).*
[2] *Tingming Wang, From Proof of inequality of matrix-rank by the block matrix construction[J], Higher mathematics(2008).*
[3] *Wajin Zhuang , Guaidance of matrix theory on skew field[M], Science press, Beijing(2006).*

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