A Note on Special Abel Matrices into $\lambda$

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Abstract. In [1], Friday introduced and studied the Abel Matrix $A_t$ as mappings into $\ell$. If $t$ is a sequence in $(0,1)$ that converges to 1, then the $A_t$ Abel Matrix [1] is defined by $a_{nk} = t_n^k \left(1 - t_n\right)$. In this note, we study the special form of the $A_t$ matrix denoted by $A_v$, where $v_n = 1 - \left(\frac{1}{n+2}\right)^q$, $q > 1$. The matrix $A_v$ is defined by

$$a_{nk} = (k+1) v_n^k (1 - v_n)^2 = (k+1) \left[1 - \left(\frac{1}{n+2}\right)^q\right]^k \left(\frac{1}{n+2}\right)^{2q}$$

It will be shown that is $A_v$ an $\ell - \ell$ matrix. Also, the translativity of $A_v$ in the $\ell - \ell$ setting is investigated.

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1. Basic notations and definitions.

Let $A = (a_{nk})$ be an infinite matrix defining a sequence to a sequence summability transformation given by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k ,$$
where \( (Ax)_n \) denotes the \( n \)th term of the image sequence \( Ax \). Let \( y \) be a complex number sequence. Throughout this paper we use the following basic notations and definitions:

(i) \( c = \{ \text{the set of all convergent complex number sequences} \} \),

(ii) \( \ell = \{ y : \sum_{k=0}^{\infty} |y_k| \ \text{converges} \} \),

(iii) \( \ell (A) = \{ y : Ay \in \ell \} \),

(iv) \( c (A) = \{ y : y \ \text{is summable by} \ A \} \).

\textbf{DEFINITION 1.} If \( X \) and \( Y \) are sets of complex number sequences, then the matrix \( A \) is called an \( X-Y \) matrix if the image \( Au \) of \( u \) under the transformation \( A \) is in \( Y \) whenever \( u \) is in \( X \).

\textbf{DEFINITION 2.} The summability matrix \( A \) is said to be \( \ell - \) translative for the sequence \( u \) in \( \ell (A) \) provided that each of the sequences \( u_T \) and \( u_S \) is in \( \ell (A) \), where \( T_u = \{ u_1, u_2, u_3, \ldots \} \) and \( S_u = \{ 0, u_0, u_1, u_2, u_3, \ldots \} \).

\section{The Main Results}

\textbf{Theorem 1.} \( A_v \) is an \( \ell - \ell \) matrix.

\textbf{Proof.} We will show that the Knopp-Lorentz Theorem is satisfied.

Note that:

\[
\sum_{n=0}^{\infty} |a_{nk}| = \sum_{n=0}^{\infty} \left( k + 1 \right) v^n \left( 1 - v^n \right)^2
\]

\[
= (k + 1) \sum_{n=0}^{\infty} \left( 1 - \left( \frac{1}{n + 2} \right)^q \right)^k \left( \frac{1}{n + 2} \right)^{2q}
\]

\[
= (k + 1) \sum_{n=0}^{\infty} \left( (n + 2)^q - 1 \right)^k (n + 2)^{-q(k+2)}
\]

\[
\leq M(k+1) \int_{0}^{\infty} \left( x+2 \right)^q -1 \left( x+2 \right)^{-q(k+2)} \, dx
\]

For some \( M > 0 \). This is possible as both the summation and the integral are finite. Now, we define
where \( q = \frac{1}{p}, \ 0 < p < 1 \). By letting \( 2^p = R \) and \( \frac{R - 1}{R} = S \) and using integration by parts repeatedly we can easily deduce that

\[
g(k) = \frac{p(R-1)^k R^{-(k+2-p)}}{k+2-p} + \frac{pk(R-1)^{k-1} R^{-(k+1-p)}}{(k+2-p)(k+1-p)} + \ldots + \frac{pk(k-1)(k-2)\ldots R^{-(2-p)}}{(k+2-p)(k+1-p)(k-p)\ldots(2-p)}
\]

Using the hypothesis that \( q > 1 \), it follows that

\[
g(k) \leq \frac{(R-1)^{k+1} R^{-(k+1)}}{k+1} + \frac{k(R-1)^k R^{-k}}{(k+1)(k)} + \ldots + \frac{k(k-1)(k-2)\ldots(1-R)R^{-1}}{(k+1)(k)(k-1)\ldots1}
\]

By writing the right-hand side of the preceding inequality using the summation notation, we obtain

\[
g(k) \leq \frac{S^{k+1}}{k+1} + \frac{kS^k}{(k+1)k} + \ldots + \frac{k(k-1)\ldots S}{(k+1)k(k-1)\ldots1}
\]

\[
= \frac{S^{k+1}}{k+1} + \frac{S^k}{k+1} + \ldots + \frac{S}{k+1}
\]

By writing the right-hand side of the preceding inequality using the summation notation, we obtain

\[
g(k) \leq \frac{S}{(k+1)} \sum_{i=0}^{k} S^i.
\]

\[
\leq \frac{S}{(k+1)} \sum_{i=0}^{\infty} S^i.
\]

\[
= \frac{S}{(k+1)(1-S)} = \frac{S}{(k+1)(1-S)}.
\]
Consequently, we get
\[ \sum_{n=0}^{\infty} |a_{nk}| \leq M(k+1)g(k) \leq \frac{MS(k+1)}{(k+1)(1-S)} = \frac{MS}{1-S} \]

Hence, \( \sup_k \left\{ \sum_{n=0}^{\infty} |a_{nk}| \right\} < \infty \) and by Knopp-Lorentz Theorem [2], \( A_v \) is an \( \ell - \ell \) matrix.

**Remark 1.** If \( v \) is as defined above, then \( \arcsin(1 - v) \in \ell \).

Note that for \( 0 < x < 1 \), we have \( \arcsin x < \frac{x}{\sqrt{1-x^2}} \). Now replacing \( x \) by \( (1 - v_n) \) we get
\[ \arcsin (1 - v_n) < \frac{(1 - v_n)}{\sqrt{1 - (1 - v_n)^2}} \cdot \]

It is easy to observe that \( \arcsin(1 - v) \in \ell \) as \( (1 - v) \in \ell \).

**Remark 2.** \( \ell (A_v) \) contains unbound sequence. To see this let
\[ x_k = (-1)^k \frac{k+2}{2} \cdot \]

Then we have
\[ \left| (A_v x)_n \right| = (1 - v_n)^2 \sum_{k=0}^{\infty} (k+1)(-1)^k \frac{k+2}{2} v_n^k \]
\[ = (1 - v_n)^2 \left| \sum_{k=0}^{\infty} (-1)^k (k+1) \frac{k+2}{2} v_n^k \right| \]
\[ = (1 - v_n)^2 \left( 1 + v_n \right)^{-3} \]
\[ < (1 - v_n)^2 \left( \frac{1}{n+2} \right)^{2q} \leq \left( \frac{1}{n} \right)^{2q} . \]
Hence \((1 - v)^2 \in \ell\) and this implies that \(x \in \ell \left(A_v\right)\).

**Theorem 2.** Every \(A_v\) matrix is \(\ell\)-translative for those sequences \(x \in \ell \left(A_v\right)\) for which \(\left\{\frac{x_k}{k}\right\} \in \ell, k = 1, 2, 3, \ldots\).

**Proof.** Suppose that \(x\) is a sequence in \(\ell \left(A_v\right)\) for which \(\left\{\frac{x_k}{k}\right\} \in \ell\). We show that

1. \(T_x \in \ell \left(A_v\right),\) and
2. \(S_x \in \ell \left(A_v\right),\) where \(T_x\) and \(S_x\) are as defined in Definition 2. Let us first show that (1) holds.

Note that

\[
\left|\left(AT_x\right)_n\right| = \left(1 - v_n\right)^2 \left|\sum_{k=0}^{\infty} (k+1)x_{k+1} \left(v_n\right)^k\right|
\]

\[
= \left(1 - v_n\right)^2 \left(\frac{1}{v_n}\right) \left|\sum_{k=1}^{\infty} kx_k \left(v_n\right)^k\right|
\]

\[
= \left(1 - v_n\right)^2 \left(\frac{1}{v_n}\right) \left|\sum_{k=1}^{\infty} (k+1)x_k \left(v_n\right)^k \left(\frac{k}{k+1}\right)\right|
\]

\[
= \left(1 - v_n\right)^2 \left(\frac{1}{v_n}\right) \left|\sum_{k=1}^{\infty} (k+1)x_k \left(v_n\right)^k \left(1 - \frac{1}{k+1}\right)\right|
\]

\[
\leq A_n + B_n
\]

where

\[
A_n = \left(1 - v_n\right)^2 \left(\frac{1}{v_n}\right) \left|\sum_{k=1}^{\infty} (k+1)x_k \left(v_n\right)^k\right|
\]
and

\[ B_n = (1 - v_n) \left( \frac{1}{v_n} \right) \left| \sum_{k=1}^{\infty} \left( k+1 \right) \left( \frac{x_k}{k+1} \right) \left( v_n \right)^k \right| \]

Now if we show that both \( A \) and \( B \) are in \( \ell \), then (1) holds. But the condition that \( A \in \ell \) and \( B \in \ell \) follow easily from the hypotheses that \( x \in \ell (A_x) \) and \( \left\{ \frac{x}{k+1} \right\} \in \ell \), respectively. Next, we show that (2) holds as follows. We have

\[
\left| (AS_x)_n \right| = (1 - v_n)^2 \left| \sum_{k=0}^{\infty} (k+1)x_k \left( v_n \right)^{k+1} \right|
\]

\[
= (1 - v_n)^2 \left| \sum_{k=0}^{\infty} (k+1)x_k \left( v_n \right)^{k+1} \left( \frac{k+2}{k+1} \right) \right|
\]

\[
= (1 - v_n)^2 \left| \sum_{k=0}^{\infty} (k+1)x_k \left( v_n \right)^{k+1} \left( 1 + \frac{1}{k+1} \right) \right|
\]

\[
\leq E_n + F_n
\]

where

\[
E_n = (1 - v_n)^2 \left| \sum_{k=0}^{\infty} (k+1)x_k \left( v_n \right)^{k+1} \right|
\]

and

\[
F_n = (1 - v_n)^2 \left| \sum_{k=0}^{\infty} (k+1)x_k \left( v_n \right)^{k+1} \right|
\]

If we show that \( E \) and \( F \) are in \( \ell \), then (2) holds. But the hypotheses that \( x \in \ell (A_x) \) and \( \left\{ \frac{x}{k+1} \right\} \in \ell \) implies that both \( E \) and \( F \) are in \( \ell \), respectively, and hence the theorem follows.
Remark III. The sequence \( x \) defined by \( x_k = \frac{(-1)^k}{k} \) is one of the sequences which satisfies the condition of Theorem 2.

Notation. Suppose \( u \) is a complex number sequence such that

\[
\sum_{k=1}^{\infty} (k+1)x_kv^k < \infty
\]

and let

\[
H_u = \left\{ u : (1-v)^2 \sum_{k=1}^{\infty} (k+1)x_kv^k \rightarrow L(\text{finite}) \text{ as } v \rightarrow 1^- \right\}
\]

Theorem 3. Every \( A_v \) matrix is \( \ell - \text{translative} \) for every sequence \( u \) in \( H_u \cap \ell \left( A_v \right) \).

Proof. Let \( u \in H_u \cap \ell \left( A_v \right) \). Then we will show that

1. \( T_u \in \ell \left( A_v \right) \), and
2. \( S_u \in \ell \left( A_v \right) \), where \( T_u \) and \( S_u \) are as defined in Definition 2. Let us first show that (1) holds.

Note that

\[
\left| (AT_u)_n \right| = (1-v_n)^2 \left| \sum_{k=0}^{\infty} (k+1) u_{k+1} \left( v_n \right)^k \right|
\]

\[
= (1-v_n)^2 \left( \frac{1}{v_n} \right) \left| \sum_{k=0}^{\infty} (k+1) u_{k+1} \left( v_n \right)^{k+1} \right|
\]

\[
= (1-v_n)^2 \left( \frac{1}{v_n} \right) \left| \sum_{k=1}^{\infty} (k+1) u_k \left( v_n \right)^{k} \left( 1 - \frac{1}{k+1} \right) \right|
\]
\[ \leq A_n + B_n \]

where

\[ A_n = (1 - v_n)^2 \left( \frac{1}{v_n} \right) \left| \sum_{k=1}^{\infty} (k+1) u_k (v_n)^k \right| \]

and

\[ B_n = (1 - v_n)^2 \left( \frac{1}{v_n} \right) \left| \sum_{k=1}^{\infty} (k+1) \left( \frac{u_k}{k+1} \right) (v_n)^k \right| \]

Now if show that both \( A \) and \( B \) are in \( \ell \), then \( (1) \) is proved. But the hypothesis that \( u \in \ell(A) \) implies that \( A \in \ell \) and \( B \in \ell \) will be shown as follows. Note that

\[ B_n = (1 - v_n)^2 \left( \frac{1}{v_n} \right) \left| \sum_{k=1}^{\infty} (k+1) \left( \frac{u_k}{k+1} \right) (v_n)^k \right| \]

\[ = \frac{(1 - v_n)^2}{(v_n)^2} \left| \sum_{k=1}^{\infty} (k+1) u_k \left( \int_0^{v_n} v^k dv \right) \right| \]

\[ = \frac{(1 - v_n)^2}{(v_n)^2} \left| \int_0^{v_n} dv \sum_{k=1}^{\infty} (k+1) u_k v^k \right| \]

The interchanging of the integral and the summation is legitimate as the radius of convergence of the power series \( \sum_{k=1}^{\infty} (k+1) u_k v^k \) is at least 1 and the power series converges absolutely and uniformly \( 0 \leq v \leq v_n \).
Now let
\[ g(v) = \sum_{k=1}^{\infty} (k+1) u_k v^k. \]

Then we have
\[ g(v)(1-v)^2 = (1-v)^2 \sum_{k=1}^{\infty} (k+1) u_k v^k \]
and the assumption that

(i) \[ u \in H_v \Rightarrow \lim_{v \to 0^+} g(v)(1-v)^2 = L < \infty \quad \text{for } v \in (0,1) \]

We also have

(ii) \[ \lim_{v \to 0} g(v)(1-v)^2 = 0 \]

Thus from (i) and (ii), it follows that
\[ \left| g(v)(1-v)^2 \right| \leq M_1, \quad \text{for some } M_1 > 0 \]

Hence we have
\[ \left| g(v) \right| \leq M_1 (1-v)^{-2} \]

Consequently, we have
\[ B_n = \left( \frac{1-v_n}{v_n} \right)^2 \left| \int_0^{v_n} g(v) dv \right| \]
\[ \leq M_2 (1-v_n)^2 \int_0^{v_n} |g(v)| dv \]
\[ \leq M_1 M_2 (1-v_n)^2 \int_0^{v_n} (1-v)^{-2} dv \]
\[= M_1 M_2 (1 - v_n) - M_1 M_2 (1 - v_n)^2\]

\[\leq 2 M_1 M_2 (1 - v_n)\]

\[= 2 M_1 M_2 \left(\frac{1}{n + 2}\right)^q\]

\[\leq 2 M_1 M_2 \left(\frac{1}{n}\right)^q\]

So, we have \(B \in \ell\) and hence (1) is proved. Next we show that (2) holds. We have

\[
\left| (AS_n)_n \right| = (1 - v_n)^2 \left| \sum_{k=1}^{\infty} (k + 1)x_{k-1} (v_n)^k \right|
\]

\[
= (1 - v_n)^2 \left| \sum_{k=0}^{\infty} (k + 2)x_k (v_n)^{k+1} \right|
\]

\[
= (1 - v_n)^2 \left| \sum_{k=0}^{\infty} (k + 1)x_k (v_n)^{k+1} \left( \frac{k + 2}{k + 1} \right) \right|
\]

\[
= (1 - v_n)^2 \left| \sum_{k=0}^{\infty} (k + 1)x_k (v_n)^{k+1} \left( 1 + \frac{1}{k + 1} \right) \right|
\]

\[\leq E_n + F_n\]

where

\[E_n = (1 - v_n)^2 \left| \sum_{k=0}^{\infty} (k + 1)x_k (v_n)^{k+1} \right|
\]

and

\[F_n = (1 - v_n)^2 \left| \sum_{k=0}^{\infty} (k + 1) \left( \frac{x_k}{k + 1} \right) (v_n)^{k+1} \right|
\]
Now if show that both \( E \) and \( F \) are in \( \ell \), then (2) is proved. But the hypothesis that \( u \in \ell(A_v) \) implies that \( E \in \ell \) and \( F \in \ell \) can be easily shown using the same technique applied in showing \( B \in \ell \) in (1). Hence the theorem follows.

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**References**


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