Alcoves of the Non-Weyl Group $H_4$

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Abstract. In [5], Shi defines the alcove form of the elements in the affine Weyl groups. In this paper, we define the alcoves of the elements in the non-Weyl group $H_4$, and we get some results similar to those in [5], then we give some examples in the parabolic subgroup $H_3$ of $H_4$. We also get that each left cell in $H_3$ is left-connected which verify the conjecture (given by Lusztig in [2]) in the case of non-crystallographic Coxeter group.

Keywords: The Coxeter group $H_4$, the positive root system, the alcoves, left cells, left-connected

1. Notation and notion

Let $(W, S)$ be the Coxeter system with $S$ its Coxeter generator set of rank $n$, i.e. $S = \{s_1, s_2, \ldots, s_n\}$, with $m(i, j) = o(s_is_j)$ (the order of $s_is_j$ in $W$) for some $s_i, s_j \in S$. For $w \in W$, we denote by $l(w)$ the length of $w$.

Let ” $\leq$ ” be the Bruhat ordering of the Coxeter group $W$ which means that for $x, y \in W, x \leq y \iff$ if $y = s_1s_2 \ldots s_r$ is reduced, then $x = s_{i_1}s_{i_2} \ldots s_{i_q}$ for $\{i_1, i_2, \ldots, i_q\}$ is a subsequence of $\{1, 2, \ldots, r\}$.

Let $E$ be the euclidean space spanned by an irreducible root system $\Phi$ of type $W$ consisting of unit vectors. Let $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the simple root system corresponding to $S$, $\Phi^+$ be the corresponding positive root system. It is well known that for $\alpha, \beta \in \Delta$, the inner product $(\alpha, \beta) = \cos \frac{\pi}{m}$, where $m = o(s_\alpha s_\beta), s_\alpha$ (resp. $s_\beta$) is the reflection determined by $\alpha$ (resp. $\beta$).

For $w \in W$, we associate two subsets of $S$ as below.

$$L(w) = \{s \in S | sw < w\}, \quad R(w) = \{s \in S | ws < w\}. $$
Clearly, \( \mathcal{L}(w) = \mathcal{R}(w^{-1}) \).

3 **Lemma 1.1:** (see [4]) For any Coxeter system \((W, S)\), to each \(s \in S\), we have \(s(\alpha_s) = -\alpha_s\) and \(s(\Phi^+) = \Phi^+ \setminus \{\alpha_s\}\).

By above lemma, we know that each simple reflection permutes the positive root system except the corresponding simple root.

**Lemma 1.2:** (see [5]) Suppose that \(\{k(w, \alpha)|\alpha \in \Phi^+\}\) is the corresponding alcove for \(w\) in an affine weyl group \(W\). then

1. \(l(w) = \sum_{\alpha \in \Phi^+} |k(w, \alpha)|\), where the notation \(|x|\) stands for the absolute value of \(x\);
2. \(\mathcal{R}(w) = \{s \in S | k(w, \alpha_s) = -1\}\).

Let \(H\) be the Hecke algebra of the Coxeter group \(W\) over \(A = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]\) (the Laurent polynomial ring in \(\mathbb{Z}\)), with a standard \(A\)–basis \(\{T_w | w \in W\}\). In [3], Kazhdan and Lusztig define another \(A\)–basis \(\{C_w | w \in W\}\) for \(H\) with

\[
C_w = \sum_{x \in w} (-1)^{l(w) - l(x)} q^{(1/2)l(w) - l(x)} P_{x, w}(q^{-1}) T_x
\]

where \(P_{w, w} = 1\) and \(P_{x, w} \in \mathbb{Z}[q]\) has degree \(\leq \frac{1}{2}(l(w) - l(x) - 1)\) if \(x < w\), \(P_{x, w} = 0\), if \(x \notin w\).

The polynomials \(P_{x, w}\) are called Kazhdan–Lusztig polynomials. Also in [3], Kazhdan and Lusztig define several relations on \(W\):

- \(x \prec w\) if \(x < w\), and degree \(P_{x, w} = \frac{1}{2}(l(w) - l(x) - 1)\); \(x \prec w\) if \(x < w\) or \(w < x\);
- \(x \leq_L w\) if there is a sequence \(x = x_0, x_1, ..., x_n = w\), with \(x_{i-1} \prec x_i\) and \(\mathcal{L}(x_{i-1}) \notin \mathcal{L}(x_i)\) for \(1 \leq i \leq n\);
- \(x \leq_R w\) if there is a sequence \(x = x_0, x_1, ..., x_n = w\), with \(x_{i-1} \prec x_i\) and \(\mathcal{R}(x_{i-1}) \notin \mathcal{R}(x_i)\) for \(1 \leq i \leq n\);
- \(x \leq_{LR} w\) if there is a sequence \(x = x_0, x_1, ..., x_n = w\), with \(x_{i-1} \leq_L x_i\) or \(x_{i-1} \leq_R x_i\) for \(1 \leq i \leq n\).

The relation \(\leq_L\) is a preorder on \(W\) whose equivalence classes are the left cells of \(W\). That is, \(x, y \in W\) are in the same left cell, (denoted by \(x \sim_L y\)) if and only if \(x \leq_L y \leq_L x\). The right (resp. two-sided) cells of \(W\) are defined in the same way, with the preorder \(\leq_R\) (resp. \(\leq_{LR}\)) replacing \(\leq_L\).

**Lemma 1.3:** (see [3]) if \(x \leq_L y\), then \(\mathcal{R}(x) \supset \mathcal{R}(y)\). Thus, if \(x \sim_L y\), then \(\mathcal{R}(x) = \mathcal{R}(y)\).
2. THE ALCOVES IN THE COXETER GROUP $H_4$

In this and next sections, $W$ always denotes the Coxeter group $H_4$, $S = \{s_1, s_2, s_3, s_4\}$ with $(s_1 s_2)^5 = (s_1 s_3)^2 = (s_1 s_4)^2 = (s_2 s_3)^2 = (s_2 s_4)^2 = (s_3 s_4)^3 = 1$.

Obviously, the Coxeter group $H_3$ is a parabolic subgroup of $H_4$ with generators set $S' = \{s_1, s_2, s_3\} \subseteq S$.

Let $\Delta = \{\alpha s_1, \alpha s_2, \alpha s_3, \alpha s_4\}$ (resp. $\Delta' = \{\alpha s_1, \alpha s_2, \alpha s_3\}$) be the simple root system corresponding to $S$ (resp. $S'$). Let $\Phi$ (resp. $\Phi'$) be the root system of $H_4$ (resp. $H_3$) and $\Phi^+$ (resp. $\Phi'^+$) be the positive root system corresponding to $\Delta$ (resp. $\Delta'$). It is well known that $|\Phi^+| = 60$, $|\Phi'^+| = 15$, where $|\cdot|$ means the cardinality of a set.

We define an alcove corresponding to $w \in W$ as a $\Phi^+$-tuple $\{k(\alpha, w)|\alpha \in \Phi^+\}$ in $\mathbb{Z}$.

If $w = 1$ (the identity element in $W$), we define $k(\alpha, w) = 0$ for all $\alpha \in \Phi^+$. And we also define the operators $s$ for $s \in S$ as following.

$$s: \quad k(\alpha, w) \mapsto k(\alpha, ws)$$

$$k(\alpha, ws) = \begin{cases} 
  k(s(\alpha), w), & \alpha \neq \alpha_s; \\
  -k(\alpha, w) - 1, & \alpha = \alpha_s. 
\end{cases}$$

By recurrence, for any $w \in W$, $w$ is corresponding one-to-one an alcove of $\{k(\alpha, w)|\alpha \in \Phi^+\}$.

For example, if $w = s \in S$ in $W$, the corresponding general alcove is $\{k(\alpha, w)|\alpha \in \Phi^+\}$ with $k(\alpha_s, w) = -1$ and $k(\alpha, w) = 0$ for all $\alpha \in \Phi^+ \setminus \{\alpha_s\}$.

3. THE RESULTS FOR THE ALCOVES IN $H_4$

For $w \in W$, $\{k(\alpha, w)|\alpha \in \Phi^+\}$ is the corresponding alcove of $w$. We have the following results.

**Proposition 3.1:** For each $w \in W$, $k(\alpha, w) = \begin{cases} 
  0; & \text{For all } \alpha \in \Phi^+. \\
  -1. & \text{For } \alpha = \alpha_s. 
\end{cases}$

It is obvious by the definition of the alcove in above section.

**Theorem 3.2:** For $w \in W$, $\{k(\alpha, w)|\alpha \in \Phi^+\}$ is the corresponding alcove of $w$. Then we have

1. $l(w) = \Sigma_{\alpha \in \Phi^+} |k(\alpha, w)|$;
2. $R(w) = \{s \in S|k(\alpha_s, w) = -1\}$.

**Proof:** Proceed by induction on $l(w) = r$. 
If \( l(w) = 0 \), it is trivial. If \( l(w) = 1 \), then \( w = s \) for some \( s \in S \), the results follows by the remark of above section. Now suppose \( l(w) > 1 \), we can write \( w = w_1s \) for some \( s \in S \) and \( l(w) = l(w_1) + 1 \).

By induction, \( l(w_1) = \sum_{\alpha \in \Phi^+} k(\alpha, w_1) \); \( \mathcal{R}(w_1) = \{ s \in S | k(\alpha, w_1) = -1 \} \).

By Proposition 3.1, \( k(\alpha_s, w_1) = 0 \) since \( s \notin \mathcal{R}(w_1) \). Then by the definition of the alcove, we have \( k(\alpha_s, w) = -1 \) and \( \sum_{\alpha \in \Phi^+ \setminus \{ \alpha_s \}} k(\alpha, w) = \sum_{\alpha \in \Phi^+ \setminus \{ \alpha_s \}} k(\alpha, w_1) \).

So \( \sum_{\alpha \in \Phi^+} k(\alpha, w) = \sum_{\alpha \in \Phi^+} k(\alpha, w_1) + 1 = l(w_1) + 1 \). Thus (1) is true for all \( w \in W \).

If \( s' \in \mathcal{R}(w_1) \), i.e. \( k(\alpha_{s'}, w_1) = -1 \). Two cases are possible.

In case \( s's = ss' \), clearly \( s' \in \mathcal{R}(w) \) and \( k(\alpha_{s'}, w) = k(\alpha_{s'}, w_1) = -1 \) by the definition.

In case \( s's \neq ss' \), we can consider \( k(\alpha, w) \) as \( w \) is in the dihedral group generated by \( \{ s, s' \} \). If \( s' \in \mathcal{R}(w) \), so \( k(\alpha_{s'}, w) = -1 \). And suppose \( s' \notin \mathcal{R}(w) \), so \( k(\alpha_{s'}, w) = 0 \).

Thus (2) follows.

**Corollary 3.3:** Let \( w_0 \) be the longest element in \( W \), then \( k(\alpha, w_0) = -1 \) for all \( \alpha \in \Phi^+ \).

By Theorem 3.2(1), Proposition 3.1 and \( l(w_0) = |\Phi^+| \), it is true.

4. **The Example:** \( H_3 \)

In this and next sections, \( W \) always denotes the Coxeter group \( H_3 \). In this section, we give the explanations in \( W \) as examples. It is well known that \( S' = \{ s_1, s_2, s_3 \} \subseteq S \). Let \( \Delta' = \{ \alpha_{s_1}, \alpha_{s_2}, \alpha_{s_3} \} \) be the simple root system corresponding to \( S' \). Set \( b := \cos \frac{\pi}{5} = \frac{\sqrt{5} + 1}{4} \), then \( 4b^2 = 2b + 1 \).

We arrange these positive roots as a \( 5 \times 3 \)-matrix for practice.

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 \\
\alpha_4 & \alpha_5 & \alpha_6 \\
\alpha_7 & \alpha_8 & \alpha_9 \\
\alpha_{10} & \alpha_{11} & \alpha_{12} \\
\alpha_{13} & \alpha_{14} & \alpha_{15}
\end{pmatrix}
\]

where \( \alpha_1 = \alpha_{s_1}, \alpha_2 = \alpha_{s_2}, \alpha_3 = \alpha_{s_3}, \alpha_4 = 2b\alpha_1 + \alpha_2, \alpha_5 = \alpha_1 + 2b\alpha_2, \alpha_6 = \alpha_2 + \alpha_3, \alpha_7 = 2b(\alpha_1 + \alpha_2), \alpha_8 = 2b\alpha_1 + \alpha_2 + \alpha_3, \alpha_9 = \alpha_1 + 2b(\alpha_2 + \alpha_3), \alpha_{10} = 2b(\alpha_1 + \alpha_2 + \alpha_3), \alpha_{11} = 2b\alpha_1 + (2b + 1)\alpha_2 + \alpha_3, \alpha_{12} = 2b\alpha_1 + (2b + 1)\alpha_2 + 2b\alpha_3, \alpha_{13} = (2b + 1)(\alpha_1 + \alpha_2) + \alpha_3, \alpha_{14} = (2b + 1)(\alpha_1 + \alpha_2) + 2b\alpha_3, \alpha_{15} = (2b + 1)\alpha_1 + 4b\alpha_2 + 2b\alpha_3. \)
By the definition, each simple reflection acts on the general alcove as following.

\[ s_1 : \begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 \\
\alpha_4 & \alpha_5 & \alpha_6 \\
\alpha_7 & \alpha_8 & \alpha_9 \\
\alpha_{10} & \alpha_{11} & \alpha_{12} \\
\alpha_{13} & \alpha_{14} & \alpha_{15}
\end{pmatrix}
\mapsto\begin{pmatrix}
-\alpha_1 - 1 & \alpha_4 & * \\
\alpha_2 & \alpha_7 & \alpha_8 \\
\alpha_5 & \alpha_6 & \alpha_{10} \\
\alpha_9 & \alpha_{13} & \alpha_{14} \\
\alpha_{11} & \alpha_{12} & *
\end{pmatrix}.\]

\[ s_2 : \begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 \\
\alpha_4 & \alpha_5 & \alpha_6 \\
\alpha_7 & \alpha_8 & \alpha_9 \\
\alpha_{10} & \alpha_{11} & \alpha_{12} \\
\alpha_{13} & \alpha_{14} & \alpha_{15}
\end{pmatrix}
\mapsto\begin{pmatrix}
\alpha_5 & -\alpha_2 - 1 & \alpha_6 \\
\alpha_7 & \alpha_1 & \alpha_3 \\
\alpha_4 & \alpha_{11} & * \\
\alpha_{12} & \alpha_8 & \alpha_{10} \\
* & \alpha_{15} & \alpha_{14}
\end{pmatrix}.\]

\[ s_3 : \begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 \\
\alpha_4 & \alpha_5 & \alpha_6 \\
\alpha_7 & \alpha_8 & \alpha_9 \\
\alpha_{10} & \alpha_{11} & \alpha_{12} \\
\alpha_{13} & \alpha_{14} & \alpha_{15}
\end{pmatrix}
\mapsto\begin{pmatrix}
* & \alpha_6 & -\alpha_3 - 1 \\
\alpha_8 & \alpha_9 & \alpha_2 \\
\alpha_{10} & \alpha_4 & \alpha_5 \\
\alpha_7 & \alpha_{12} & \alpha_{11} \\
\alpha_{14} & \alpha_{13} & *
\end{pmatrix}.\]

where the entries in the * position unchanged.

So, some elements in \( H_3 \) and their corresponding alcove are listed as below.

\[ 1 \leftrightarrow \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} ;

s_1 \leftrightarrow \begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} ;

s_2 \leftrightarrow \begin{pmatrix}
0 & -1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} ;

s_3 \leftrightarrow \begin{pmatrix}
0 & 0 & -1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} ;

s_1s_2 \leftrightarrow \begin{pmatrix}
0 & -1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} ;

s_2s_1 \leftrightarrow \begin{pmatrix}
-1 & 0 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} ;

w_0 \leftrightarrow \begin{pmatrix}
-1 & -1 & -1 \\
-1 & -1 & -1 \\
-1 & -1 & -1 \\
-1 & -1 & -1
\end{pmatrix} ;
where 1 is the identity element and \( w_0 \) is the longest element in \( W \).

5. The cells in the Coxeter group \( H_3 \)

In [1], D. Alvis gives the left cells and two-sided cells of the Coxeter group \( H_4 \) by computing the Kazhdan-Lusztig polynomials. In this paper, we give the left cells by using the general alcoves in \( H_3 \). From Lemma 1.3, we can give the left cells in the set \( W_J = \{ w \in W | \mathcal{R}(w) = J \} \) with \( J \subseteq S' \), \( X^* = w_0 \cdot X \) where \( w_0 \) is the longest element in \( W \).

First, definite \( W_0 := W_{\varnothing} \)

\[
W_1^{(1)} = \{1, 21, 121, 321, 2121, 32121\}, \\
W_1^{(2)} = \{3121, 23121, 123121, 2123121, 32123121\}, \\
W_1^{(3)} = \{232121, 1232121, 21232121, 121232121\}, \\
W_1^{(4)} = \{321232121, 1321232121, 21321232121, 121321232121\}; \\
W_2^{(1)} = \{2, 12, 32, 212, 1212, 3212\}, W_2^{(2)} = \{312, 2312, 12312, 212312, 3212312\}, \\
W_2^{(3)} = \{31212, 231212, 1231212, 21231212, 321231212\}, \\
W_2^{(4)} = \{2321212, 12321212, 212321212, 1212321212\}, \\
W_2^{(5)} = \{3212321212, 13212321212, 213212321212, 1213212321212\}, \\
W_2^{(6)} = \{121231212, 3121231212, 2131231212, 321231231212, 212312312312\}; \\
W_3^{(1)} = \{3, 23, 123, 2123, 12123, 32123\}, \\
W_3^{(2)} = \{312123, 2312123, 12312123, 2123123123, 3212312123\},
\]

where for the sake of simplifying the notation, denote by \( i \) the reflection \( s_i (i = 1, 2, 3) \).

Obviously, \( W_{(1)} = \cup_{i=1}^{2} W_1^{(i)}, \ W_{(2)} = \cup_{i=1}^{6} W_2^{(i)}, \ W_{(3)} = \cup_{i=1}^{2} W_3^{(i)}, \ W_{(1,2)} = W^{*} \), \( W_{(1,3)} = W_{*}^{*} \), \( W_{(2,3)} = W_{*}^{*} \), \( W_{(1,2,3)} = W_{*}^{*} \).

we have

**Theorem 5.1:** The left cells of \( W \) are the subsets \( W_i^{(j)} \), \( (W_i^{(j)})^* \). Thus, there are 26 left cells in \( W \).

A subset \( K \) of any Coxeter group \( W \) is left-connected (resp., right-connected), if for any \( x, y \in K \), there exists a sequence of elements \( x_0 = x, x_1, \ldots, x_r = y \) in \( K \) with some \( r \geq 0 \) such that \( x_{i-1}x_i^{-1} \in S \) (resp., \( x_i^{-1}x_{i+1} \in S \) for \( 1 \leq i \leq r \). Lusztig conjectured in [2] that if \( W \) is an affine Weyl group then any left cell \( L \) of \( W \) is left-connected. In [6], Shi and author verify the conjecture in the case of the left cells with \( a \)-value 4 in the affine Weyl groups \( \widetilde{E}_i (i = 6, 7, 8) \) and the conjecture is supported by all the existing data.
From the above construction of left cells in $H_3$, we get the following result which verify the conjecture in the case of non-Weyl group.

**Corollary 5.2:** Each left cell of $H_3$ is left-connected.

## References


[6] J. Y. Shi and X.G. Zhang, Left cells with a-value 4 in the affine weyl groups $\tilde{E}_i (i = 6, 7, 8)$, Comm. in Algebra, 36 (2008), 3317-3346;

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