International Mathematical Forum, 4, 2009, no. 32, 1573 - 1585

t-Pebbling the Product of Fan Graphs and the Product of Wheel Graphs

A. Lourdusamy

Department of Mathematics, St. Xavier's College (Autonomous)
Palayamkottai - 627 002, India
lourdugnanam@hotmail.com

S. Samuel Jeyaseelan

Department of Mathematics, St. Joseph's College (Autonomous)
Trichy – 620 002, India
samjeya@yahoo.com

A. Punitha Tharani

Department of Mathematics, St. Mary's College, Tuticorin, India
punitha_tharani@yahoo.co.in

Abstract

The t-pebbling number, $f_t(G)$, of a connected graph G, is the smallest positive integer such that from every placement of $f_t(G)$ pebbles, t pebbles can be moved to a specified target vertex by a sequence of pebbling moves, each move taking two pebbles off a vertex and placing one on an adjacent vertex. In this paper, we compute the t-pebbling number of fan graphs and wheel graphs and we study the conjecture: $f_t(G \times H) \leq f(G)\, f_t(H)$, for the product of fan graphs and for the product of wheel graphs.

Mathematics Subject Classification: 05C05

Keywords: Pebbling, Graham's Conjecture, Direct products, Graph parameters

1 Introduction

Graph pebbling is a model for the transmission of consumable resources. Chung [1] defines a pebbling distribution on a connected graph as a placement of pebbles on the vertices of the graph. A *pebbling move* then consists of removing two pebbles from one vertex, throwing one away, and putting the other pebble on an adjacent vertex. Chung defined the *pebbling number* of a *vertex* v in G as the smallest number $f(v, G)$ such that from every placement of $f(v, G)$ pebbles, it is possible to move a pebble to v by a sequence of pebbling moves. She also defined the *t-pebbling number* of v in G as the smallest number $f_t(v, G)$ such that from every placement of $f_t(v, G)$ pebbles, it is possible to move t pebbles to v. Then the t-pebbling number of G is the smallest number $f_t(G)$ such that from any placement of $f_t(G)$ pebbles, it is possible to move t pebbles to any specified target by a sequence of pebbling moves. Thus $f_t(G)$ is the largest value of $f_t(v, G)$ over all vertices v. The value of $f_t(G)$ for $t = 1$ is the *pebbling number of* G, denoted by $f(G)$.

Throughout this paper G denotes a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Also for any vertex $v \in V(G)$, $d(v)$ denotes the degree of v.

Chung also defined the *two pebbling property* of a graph, and Wang [12] extended Chung's definition to the *odd two-pebbling property.* In [7] we find the following definitions.

Definition 1.1 ([7]). *Given the t-pebbling number of G, let p be the number of pebbles on G, let q be the number of vertices with at least one pebble. We say that G satisfies the* $2t$-pebbling property *if it is possible to move $2t$ pebbles to any specified target vertex of G starting from every configuration in which $p \geq 2f_t(G) - q + 1$ or equivalently $p + q > 2f_t(G)$ for all t.*

If q stands for the number of vertices with an odd number of pebbles, we call the property, the odd $2t$-pebbling property.

Definition 1.2 ([7]). *We say a graph satisfies the* odd $2t$-pebbling property *for all t if, for any arrangement of pebbles with at least $2f_t(G) - r + 1$ pebbles, where r is the number of vertices in the arrangement with an odd number of pebbles, it is possible to put $2t$ pebbles on any target vertex using pebbling moves.* ■

It is easy to see that a graph which satisfies the $2t$-pebbling property also satisfies the odd $2t$-pebbling property for all t. For $t = 1$, Definition 1.1 gives the two pebbling property and Definition 1.2 gives the odd two-pebbling property.

With regard to the t-pebbling number of graphs, we find the following theorems in [6, 8, 9, 10].

Theorem 1.3 ([10]). *Let G be a connected graph on n vertices where $n \geq 2$. Let there be a vertex v such that $d(v) = n - 1$. Then $f_t(v, G) = 2t + n - 2$.* ■

Theorem 1.4 ([10]). *Let K_n be the complete graph on n vertices where $n \geq 2$. Then $f_t(K_n) = 2t + n - 2$.* ■

Theorem 1.5 ([6]). *Let* $K_1 = \{v\}$. *Let* $C_{n-1} = (u_1, u_2, \dots, u_{n-1})$ *be a cycle of length* $n-1$. *Then the* t*-pebbling number of the wheel graph* W_n *is* $f_t(W_n) = 4t + n - 4$ *for* $n \geq 5$. ■

Definition 1.6. *A graph* $G = (V, E)$ *is called an* r*-partite graph if* V *can be partitioned into* r *non-empty subsets* $V_1, V_2, \dots, V_r$ *such that no edge of* G *joins vertices in the same set. The sets* $V_1, V_2, \dots, V_r$ *are called* partite sets *or* vertex classes *of* G.

If G *is an* r*-partite graph having partite sets* $V_1, V_2, \dots, V_r$ *such that every vertex of* V_i *is joined to every vertex of* V_j, *where* $1 \leq i, j \leq r$ *and* $i \neq j$, *then* G *is called a* complete r-partite graph. *If* $|V_i| = s_i$, *for* $i = 1, 2, \dots, r$, *then we denote* G *by* $K_{s_1, s_2, \dots, s_r}$.

Notation 1.7 ([8]). *For* $s_1 \geq s_2 \geq \cdots \geq s_r$, $s_1 > 1$ *and if* $r = 2$, $s_2 > 1$, *let* $K^*_{s_1, s_2, \dots, s_r}$ *be the complete* r*-partite graph with* $s_1, s_2, \dots, s_r$ *vertices in vertex classes* $C_1, C_2, \dots, C_r$ *respectively. Let* $n = \sum_{r=1}^{r} s_i$.

Theorem 1.8 ([8]). *For* $G = K^*_{s_1, s_2, \dots, s_r}$,

$$f_t(G) = \begin{cases} 2t + n - 2, & \text{if } 2t \leq n - s_1 \\ 4t + s_1 - 2, & \text{if } 2t \geq n - s_1 \end{cases}$$

Theorem 1.9 ([10]). *Let* $K_{1,n}$ *be an* n*-star where* $n > 1$.
Then $f_t(K_{1,n}) = 4t + n - 2$. ■

Theorem 1.10 ([10]). *Let* C_n *denote a simple cycle with* n *vertices, where* $n \geq 3$. *Then*

$$f_t(C_n) = \begin{cases} t\,(2^{\frac{n}{2}}), & \text{if } n \text{ is even} \\ 1 + (t-1)\,(2^{\lfloor \frac{n}{2} \rfloor}) + 2\left(\left\lceil \frac{2}{3}(2^{\lfloor \frac{n}{2} \rfloor} - 1) \right\rceil \right), & \text{if } n \text{ is odd} \end{cases}$$

Theorem 1.11 ([10]). *Let* P_n *be a path on* n *vertices. Then* $f_t(P_n) = t\,(2^{n-1})$. ■

Theorem 1.12 ([10]). *Let* Q_n *be the* n*-cube. Then* $f_t(Q_n) = t\,(2^n)$. ■

With regard to the $2t$-pebbling property, we find the following theorems in [7, 8, 9, 11].

Theorem 1.13 ([11]). *All diameter two graphs satisfy the two-pebbling property.* ■

Theorem 1.14 ([7]). *All paths satisfy the* $2t$*-pebbling property for all* t. ■

Theorem 1.15 ([7]). *All even cycles satisfy the $2t$-pebbling property for all t.* ∎

Theorem 1.16 ([7]). *The n-cube Q_n satisfies the $2t$-pebbling property for all t.* ∎

Theorem 1.17 ([8]). *Let K_n be a complete graph on n vertices. Then K_n satisfies the $2t$-pebbling property for all t.* ∎

Theorem 1.18 ([9]). *The star graph $K_{1,n}$, where $n > 1$ satisfies the $2t$-pebbling property.* ∎

Theorem 1.19 ([9]). *Any complete r-partite graph satisfies the $2t$-pebbling property.* ∎

Now we find the t-pebbling numbers of fan graphs. We show that fan graphs satisfy the $2t$-pebbling property. We also give an alternate proof for the t-pebbling number of wheel graphs.

2 The t-Pebbling Number of Fan Graphs and Wheel Graphs

A fan graph, denoted by F_n, is a path P_{n-1} plus an extra vertex connected to all vertices of the path P_{n-1}. Throughout this paper, a fan graph with vertices $v_0, v_1, \ldots, v_{n-1}$ in order means the fan graph F_n whose vertices of the path P_{n-1} are $v_1, \ldots, v_{n-1}$ in order and whose extra vertex is v_0.

For any vertex v of a graph G, $p(v)$ refers to the number of pebbles on v.

We find the following theorem in [3].

Theorem 2.1. *The pebbling number of the fan graph F_n is $f(F_n) = n$.* ∎

Theorem 2.2. *Let F_n be a fan graph on n vertices in order. For $n \geq 4$, $f_t(F_n) = 4t + n - 4$.*

Proof. By Theorem 1.3, $f_t(v_0, F_n) = 2t + n - 2$.

Let us now find the t-pebbling number of v_1. Without loss of generality, we assume that v_1 has zero pebbles on it. We place $4t - 1$ pebbles on v_{n-1} and one pebble each on the vertices of F_n, other than v_0, v_1, v_2 and v_{n-1}. In this configuration of pebbles, we cannot move t pebbles to v_1. So $f_t(v_1, F_n) \geq 4t + n - 4$. Hence $f_t(F_n) \geq 4t + n - 4$ for all $t \geq 1$.

Let us now use induction on t to show that $f_t(F_n) \leq 4t + n - 4$. For $t = 1$, the theorem is true by Theorem 2.1. We now assume $t > 1$. Suppose $4t + n - 4$ pebbles are placed on the vertices of F_n. First let the target vertex be v_0. By Theorem 1.3, $f_t(v_0, F_n) = 2t + n - 2$ and this is less than $4t + n - 4$ for $t > 1$.

Next, suppose the target vertex is v_k and $p(v_k) = 0$, where $k \in \{1, \ldots, n-1\}$. We consider the following cases:
Case (i): If $p(v_0) \geq 2$, then using two pebbles we can put a pebble on v_k. Then the remaining number of pebbles on the vertices of F_n will be $4t+n-6$. Then these pebbles would suffice to put $t-1$ additional pebbles on v_k.
Case (ii): If $p(v_0) = 1$, then we can find some v_i $(i \neq k)$ with $p(v_i) \geq 2$. Then using two pebbles of v_i we can move a pebble to v_0. Now v_0 has two pebbles. So we can move a pebble to v_k. Then by induction the remaining number of pebbles will suffice to put $t-1$ additional pebbles.
Case (iii): If $p(v_0) = 0$, then we can find some $v_i(i \neq k)$ with $p(v_i) \geq 4$ or we can find at least two vertices $v_j(j \neq k)$ and $v_l(l \neq k)$ with $p(v_j) \geq 2$ and $p(v_l) \geq 2$. Suppose not. Then the total number of pebbles placed on the vertices of F_n will be at most n which is a contradiction to the total number of pebbles placed on the vertices of F_n. Hence we can find at least two vertices $v_j(j \neq k)$ and $v_l(l \neq k)$ with $p(v_j) \geq 2$ and $p(v_l) \geq 2$. Now we can put a pebble on v_k using at most four pebbles. Then there will be at least $4t+n-8$ pebbles remaining on the vertices of F_n. By induction, we can move $t-1$ additional pebbles to v_k. ■

Theorem 2.3. *Fan graphs satisfy the* $2t$*-pebbling property.*

Proof. Suppose we start with a configuration of $2f_t(F_n) - q + 1 = 2(4t+n-4) - q + 1 = 8t + 2n - 7 - q$ pebbles where q denotes the number of vertices of F_n with at least one pebble. Let us use induction on t to show that $2t$ pebbles can be moved to the target. For $t = 1$, the theorem is true by Theorem 1.13. We assume $t > 1$ and the target vertex has zero pebbles on it initially.
Case (1): Let the target vertex be v_0.

As q can be at most $n-1$, we start with at least $8t + 2n - 7 - q \geq 8t + 2n - 7 - n + 1 = 8t + n - 6$ pebbles.
Subcase (i): $n \leq 8t - 7$. In this case we start with at least $2n+1$ pebbles. We claim that there is at least one $i \in \{1, \ldots, n-1\}$ with $p(v_i) \geq 5$ or there exist at least $j, l \in \{1, \ldots, n-1\}$ with $p(v_j) \geq 3$ and $p(v_l) \geq 3$. Suppose not. Then $p(v_i) \leq 4$ and $p(v_k) \leq 2$ for every $k \neq i$. Therefore, the configuration has at most $(q-2)\,2+4 \leq 2(n-1) < 2n+1$ pebbles, which is a contradiction. So we can move two pebbles to v_0 using four pebbles without making any of the q occupied vertices empty. This leaves us with $8t+2n-11-q > 2f_{t-1}(F_n)-q+1$ pebbles that have not been moved with q occupied vertices. By induction, we can put $2(t-1)$ additional pebbles on v_0.
Subcase (ii): $n > 8t - 7$. If there exists $i \in \{1, \ldots, n-1\}$ with $p(v_i) \geq 5$ or there exist $j, k \in \{1, \ldots, n-1\}$ with $p(v_j) \geq 3$ and $p(v_k) \geq 3$, then as in Subcase (i), we can put $2t$ pebbles on v_0. Suppose not. We assume there exists $l \in \{1, \ldots, n-1\}$ with $p(v_l) = 4$. Therefore, $p(v_m) \leq 2$ for $m \neq l$. We claim that there will be at least $2t-1$ vertices with exactly two pebbles.

Suppose there are at most $2t-2$ vertices with exactly two pebbles. Then the configuration has at most $(2t-2)\,2+4+\big(q-(2t-1)\big)=2t+q+1$ pebbles. Note that $q \le n-1$. So the configuration has at most $2t+n$ pebbles. This is a contradiction to the total number of pebbles placed on F_n. So, there will be at least $2t-1$ vertices with exactly two pebbles and a vertex with four pebbles. Hence we can move $2t$ pebbles to v_0.

Case (2): Let the target vertex be v_k, where $k \in \{1, \ldots, n-1\}$.

Without loss of generality, we assume v_0 has zero pebbles on it. As q can be at most $n-2$, we start with at least $8t+n-5$ pebbles.

Subcase (iii): $n \le 8t-8$**.** Clearly the total number of pebbles on the vertices of F_n is at least $2n+3$. We claim that there is at least one $i \in \{1, \ldots, n-1\}$, $i \neq k$, with $p(v_i) \ge 9$ or there exist at least i and j, $i \neq k$, $j \neq k$ with $p(v_i) \ge 5$ and $p(v_j) \ge 5$ or there exist $\{i, j, l, m\} \subseteq \{1, \ldots, n-1\}$ such that $i \neq k$, $j \neq k$, $l \neq k$, $m \neq k$ with $p(v_i) \ge 3$, $p(v_j) \ge 3$, $p(v_l) \ge 3$ and $p(v_m) \ge 3$. Suppose not. Without loss of generality, we assume there is a vertex with eight pebbles. Therefore the other vertices will contain at most two pebbles. Then the number of pebbles in the configuration is at most $(q-1)\,2+8=2q+6 \le 2n+2$ as q is at most $n-2$. This is a contradiction to the total number of pebbles placed on the vertices of F_n. So, we can move four pebbles to v_0 and hence we can move two pebbles to v_k using eight pebbles without making any of the q occupied vertices empty. After this, we will be having $8t+2n-q-15=2f_{t-1}(F_n)-q+1$ pebbles that have not been moved with q occupied vertices. By induction, these pebbles would suffice to put $2(t-1)$ additional pebbles on v_k.

Subcase (iv): $n > 8t-8$**.** If there is a vertex with at least nine pebbles or there are two vertices with at least five pebbles each or there is a vertex with at least five pebbles and two vertices with at least three pebbles each or there are four vertices with at least three pebbles each then we apply induction to put $2t$ pebbles on v_k. Otherwise, without loss of generality we assume there is a vertex with eight pebbles. Thus, the other vertices will have at most two pebbles on them. We claim that there are at least $4t-1$ vertices with exactly two pebbles. Suppose, there are at most $4t-2$ vertices with exactly two pebbles. Then, the total number of pebbles in the configuration will be at most $(4t-2)\,2+8+\big(q-(4t-1)\big)=4t+5+q \le 4t+n+3$, where the last inequality follows because $q \le n-2$. This is a contradiction. Therefore, we have at least $4t-1$ vertices with exactly two pebbles and a vertex with eight pebbles. Hence we can move $4t$ pebbles to v_0 and then we can move $2t$ pebbles to v_k. ∎

We now give an alternate proof for Theorem 1.5, which is found in [6].

Lemma 2.4. *Let G be a spanning subgraph of H. Then $f_t(G) \ge f_t(H)$.*

Proof. We note that the t-pebbling number of a graph will never increase when additional edges are added to the graph. Hence the proof of the lemma follows. ∎

Theorem 2.5. *Let $K_1 = \{v\}$. Let $C_{n-1} = (u_1, u_2, \ldots, u_{n-1})$ be a cycle of length $n-1$. Then the t-pebbling number of W_n is $f_t(W_n) = 4t + n - 4$ for $n \geq 5$.*

Proof. By Theorem 1.3, $f_t(v, W_n) = 2t + n - 2$. Let us now find out the t-pebbling number of u_1. Assume that u_1 has zero pebbles. Place $4t - 3$ pebbles at $u_{\lceil \frac{n}{2} \rceil}$ and one pebble at every vertex of $W_n - \{u_1, u_{\lceil \frac{n}{2} \rceil}\}$. Then t pebbles cannot be moved to u_1. So $f_t(u_1, W_n) \geq 4t + n - 4$. By symmetry, $f_t(u_i, W_n) \geq 4t + n - 4$ for $i = 2, 3, \ldots, n-1$. Hence $f_t(W_n) \geq 4t + n - 4$.

Since F_n is a spanning subgraph of W_n, by Lemma 2.4, $f_t(W_n) \leq f_t(F_n)$. Now by Theorem 2.2, $f_t(W_n) \leq 4t + n - 4$. ∎

3 *t*-Pebbling the Product of Graphs

In this section we define the product of two graphs and discuss results on the t-pebbling number of direct product of two graphs. We also discuss the t-pebbling number of the product of two fan graphs and the t-pebbling number of product of two wheel graphs.

Definition 3.1 ([4]). *If $G = (V_G, E_G)$ and $H = (V_H, E_H)$ are two graphs, the direct product of G and H is the graph, whose vertex set is the Cartesian product $V_{G \times H} = V_G \times V_H = \{(x, y) : x \in V_G, y \in V_H\}$ and whose edges are given by $E_{G \times H} = \big\{\big((x, y), (x', y')\big) : x = x'$ and $(y, y') \in E_H$ or $(x, x') \in E_G$ and $y = y'\big\}$.*

We write $\{x\} \times H$ (respectively $G \times \{y\}$) for the subgraph of vertices whose projection on to V_G is the vertex x (respectively whose projection on to V_H is y). If the vertices of G are labeled x_i then for any distribution of pebbles on $G \times H$, we write p_i for the number of pebbles on $\{x_i\} \times H$ and q_i for the number of occupied vertices of $\{x_i\} \times H$.

Chung [1] attributed Conjecture 3.2 to Graham and Lourdusamy [7] extended Graham's Conjecture to the t-pebbling number of a graph as given in Conjecture 3.3.

Conjecture 3.2. *For any connected graphs G and H, the pebbling number of $G \times H$ satisfies $f(G \times H) \leq f(G)f(H)$.*

Conjecture 3.3 (Lourdusamy). *For any connected graphs G and H, the t-pebbling number of $G \times H$ satisfies $f_t(G \times H) \leq f(G)f_t(H)$ for all t.*

We take Lemma 3.4 from [4]. It describes how many pebbles we can transfer from one copy of H to an adjacent copy of H in $G \times H$. It is also called Transfer Lemma.

Lemma 3.4 (Transfer Lemma). *Let (x_i, x_j) be an edge in $G \times H$. Suppose that in $G \times H$, we have p_i pebbles occupying q_i vertices of $\{x_i\} \times H$. If $q_i - 1 \leq k \leq p_i$ and if k and p_i have the same parity then k pebbles can be retained on $\{x_i\} \times H$ while moving $\frac{p_i - k}{2}$ pebbles onto $\{x_j\} \times H$. If k and p_i have opposite parity we must leave $k+1$ pebbles on $\{x_i\} \times H$, so we can only move $\frac{p_i-(k+1)}{2}$ pebbles onto $\{x_j\} \times H$.*

In particular, we can always move at least $\dfrac{p_i - q_i}{2}$ pebbles onto $\{x_i\} \times H$.

We find the following theorems with regard to the t-pebbling number of direct product of two graphs in [7, 8, 9].

Theorem 3.5 ([7]). *Let P_m be a path on m vertices. When G satisfies the $2t$-pebbling property,*

$$f_t(P_m \times G) \leq f(P_m) f_t(G) \quad \textit{for all } t.$$

Theorem 3.6 ([7]). *Let P_m be a path on m vertices. Then*

$$f_t(P_m \times P_n) \leq t\, 2^{m+n-2} \quad \textit{for all } t.$$

Theorem 3.7 ([8]). *Let P_m be a path on m vertices and $K^*_{s_1,s_2,\ldots,s_r}$ be a complete r-partite graph. Then*

$$f_t(P_m \times K^*_{s_1,s_2,\ldots,s_r}) \leq f(P_m) f_t(K^*_{s_1,s_2,\ldots,s_r}) \quad \textit{for all } t.$$

Theorem 3.8 ([8]). *Let K_n be a complete graph on n vertices where $n \geq 2$ and let G be a graph with the $2t$-pebbling property. Then*

$$f_t(K_n \times G) \leq f(K_n) f_t(G) \quad \textit{for all } t.$$

Theorem 3.9 ([8]). *Let K_n be a complete graph on n vertices. Then*

$$f_t(K_n \times K^*_{s_1,s_2,\ldots,s_r}) \leq f(K_n) f_t(K^*_{s_1,s_2,\ldots,s_r}) \quad \textit{for all } t.$$

Theorem 3.10 ([8]). *Let K_n be a complete graph on n vertices. Then*

$$f_t(K_m \times K_n) \leq f(K_m) f_t(K_n) \quad \textit{for all } t.$$

Theorem 3.11 ([9]). *Let $K_{1,n}$ be an n-star $(n > 1)$. If G satisfies the $2t$-pebbling property then*

$$f_t(K_{1,n} \times G) \leq f(K_{1,m}) f_t(G) \quad \textit{for all } t.$$

Theorem 3.12 ([9]). *Let $K_{1,n}$ be an n-star, where $n > 1$. Then*

$$f_t(K_{1,n} \times K_{1,m}) \leq f(K_{1,n})\, f_t(K_{1,m}) \quad \textit{for all } t.$$

Theorem 3.13 ([9]). *Let $K_{s_1,2}$ be a complete bipartite graph with $s_1 \geq 2$ and G be a graph with the $2t$-pebbling property. Then*

$$f_t(K_{s_1,2} \times G) \leq f(K_{s_1,2})\, f_t(G) \quad \textit{for all } t.$$

Note that $f(K_{s_1,2}) = s_1 + 2$ [2].

Theorem 3.14 ([9]). *Let $K^*_{s_1,s_2,\ldots,s_r}$ be a complete r-partite graph with $s_1, s_2, \ldots, s_r$ vertices in vertex classes $C_1, C_2, \ldots, C_r$ respectively and G be a graph with the $2t$-pebbling property. Then*

$$f_t(K^*_{s_1,s_2,\ldots,s_r} \times G) \leq n f_t(G) \quad \textit{for all } t, \textit{ where}$$
$$n = f(K^*_{s_1,s_2,\ldots,s_r}) = s_1 + s_2 + \ldots + s_r \ [2]. \qquad \blacksquare$$

Theorem 3.15 ([9]). *Let $K^*_{s_1,s_2,\ldots,s_r}$ be a complete r-partite graph. Then*

$$f_t(K^*_{s_1,s_2,\ldots,s_r} \times K^*_{m_1,m_2,\ldots,m_n}) \leq f(K^*_{s_1,s_2,\ldots,s_r})\, f_t(K^*_{m_1,m_2,\ldots,m_n}) \quad \textit{for all } t.$$

We now discuss our results regarding the t-pebbling number of direct product of two graphs.

Corollary 3.16. *Let P_m be a path on m vertices and F_n be a fan graph on n vertices. Then*

$$f_t(P_m \times F_n) \leq f(P_m)\, f_t(F_n) \quad \textit{for all } t.$$

Corollary 3.17. *Let K_m be a complete graph on m vertices and F_n be a fan graph on n vertices. Then*

$$f_t(K_m \times F_n) \leq f(K_m)\, f_t(F_n) \quad \textit{for all } t.$$

Corollary 3.18. *Let $K_{1,m}$ be an m-star $(m > 1)$ and F_n be a fan graph on n vertices. Then*

$$f_t(K_{1,m} \times F_n) \leq f(K_{1,m})\, f_t(F_n) \quad \textit{for all } t.$$

Corollary 3.19. *Let $K_{s_1,2}$ be a complete bipartite graph with $s_1 \geq 2$ and F_n be a fan graph on n vertices. Then*

$$f_t(K_{s_1,2} \times F_n) \leq f(K_{s_1,2})\, f_t(F_n) \quad \textit{for all } t.$$

Corollary 3.20. *Let $K^*_{s_1,s_2,\dots,s_r}$ be a complete r-partite graph with $s_1, s_2, \dots, s_r$ vertices in vertex classes $C_1, C_2, \dots, C_r$ respectively and F_n be a fan graph on n vertices. Then*

$$f_t(K^*_{s_1,s_2,\dots,s_r} \times F_n) \le f(K^*_{s_1,s_2,\dots,s_r})\, f_t(F_n) \quad \textit{for all } t.$$

We now discuss Conjecture 3.3 for the product of fan graphs and for the product of wheel graphs.

Theorem 3.21. *Let F_n be a fan graph on n vertices $v_0, v_1, \dots, v_{n-1}$ in order. If G satisfies the $2t$-pebbling property, then*

$$f_t(F_n \times G) \le f(F_n)\, f_t(G) \quad \textit{for all } t.$$

Proof. For $n = 3$, $F_3 = K_3$. So the theorem is true by Theorem 3.8 for $n = 3$. It is easy to verify the theorem for $n = 4, 5$. We assume $n > 5$. Let p_i be the number of pebbles on $\{v_i\} \times G$ with q_i occupied vertices, where $i = 0, 1, 2, \dots, n-1$. Let $y \in G$.
Case (1): Suppose the target vertex is (v_0, y). If $p_0 \ge f_t(G)$, then we put t pebbles on (v_0, y). So we assume $p_0 < f_t(G)$. If there exists some $i \in \{1, 2, \dots, n-1\}$ with $\frac{p_i+q_i}{2} > f_t(G)$, then we can put $2t$ pebbles on (v_i, y) and so we can move t pebbles to (v_0, y). Otherwise, $\frac{p_i+q_i}{2} \le f_t(G)$ for $i = 1, 2, \dots, n-1$. Now we transfer $\frac{p_i-q_i}{2}$ pebbles from $\{v_i\} \times G$ to $\{v_0\} \times G$ for $i = 1, 2, \dots, n-1$. So we transfer $\sum_{i=1}^{n-1} \frac{p_i-q_i}{2}$ pebbles to $\{v_0\} \times G$. If $p_0 + \sum_{i=1}^{n-1} \frac{p_i-q_i}{2} \ge f_t(G)$, then we put t pebbles on (v_0, y). Suppose not. Then $p_0 + \sum_{i=1}^{n-1} \frac{p_i-q_i}{2} < f_t(G)$. Adding this with $\frac{p_i+q_i}{2} \le f_t(G)$ for $i = 1, 2, \dots, n-1$, we get

$$p_0 + p_1 + p_2 + \dots + p_{n-1} < n\, f_t(G).$$

Thus any distribution of pebbles from which we may not put t pebbles on (v_0, y) must begin with fewer than $n\, f_t(G)$ pebbles.
Case (2): (v_i, y) is a target vertex, where $i \in \{1,, 2, \dots, n-1\}$. Without loss of generality we assume that (v_{n-1}, y) is the target vertex. We take the n copies of G *i.e.,* $\{v_0\} \times G, \{v_1\} \times G, \dots, \{v_{n-1}\} \times G$, respectively as $G_0, G_1, \dots, G_{n-1}$.

If $p_0 + p_{n-2} + p_{n-1} \ge 3\, f_t(G)$, then we are done. So we assume that $p_0+p_{n-2}+p_{n-1} < 3\, f_t(G)$. Let $p_0+p_{n-2}+p_{n-1} = 3\,\alpha_0\, f_t(G)$, where $0 \le \alpha_0 < 1$. Let G_j contain $(k_j + \alpha_j)\, f_t(G)$ pebbles where k_j is a non-negative integer and $0 \le \alpha_j < 1$ for $j = 1, 2, \dots n-3$. Now we may assume that $\sum_{j=1}^{n-3} q_j > (n - 6 + 3\alpha_0)\, f_t(G)$. Suppose not. Then $\sum_{j=1}^{n-3} q_j \le (n - 6 + 3\alpha_0)\, f_t(G)$. Then we could move at least $\frac{(n-3\alpha_0)\, f_t(G) - (n-6+3\alpha_0)\, f_t(G)}{2} = 3(1 - \alpha_0)\, f_t(G)$ pebbles to G_0 and hence after this process, the number of pebbles on the subgraph $\left(F_n - \cup_{i=1}^{n-3}\{v_i\}\right) \times G$ will be at least $3(1 - \alpha_0)\, f_t(G) + 3\alpha_0\, f_t(G) = 3f_t(G)$

and so we can move t pebbles to the target. Hence we may assume that $\sum_{j=1}^{n-3} q_j > (n-6+3\alpha_0)\, f_t(G)$. Now let $s = \sum_{j=0}^{n-3} \alpha_j$. Then $s \le n-3$. It is easy to see that $\sum_{j=1}^{n-3} k_j = n - s$. Note that $\alpha_j f_t(G) + q_j < 4\, f_t(G)$ for $1 \le j \le n-3$. We claim that there exists $j_1, j_2, \ldots, j_s$ such that $j_i \ge 1$ and $\alpha_{j_i} f_t(G) + q_{j_i} > f_t(G)$, $i = 1, 2, \ldots, s$. Suppose not. Then

$$\sum_{j=1}^{n-3} \big(\alpha_j\, f_t(G) + q_j\big) < \big((n-3)-(s-1)\big)\, f_t(G) + 4(s-1)\, f_t(G).$$

$$\text{So} \quad \sum_{j=1}^{n-3} \big(\alpha_j\, f_t(G) + q_j\big) < \big((n-6)\big)\, f_t(G) + 3s f_t(G).$$

$$\text{But} \quad \sum_{j=1}^{n-3} \big(\alpha_j\, f_t(G) + q_j\big) > \sum_{j=1}^{n-3} \alpha_j\, f_t(G) + (n-6+3\alpha_0)\, f_t(G).$$

$$\text{That is} \quad \sum_{j=1}^{n-3} \big(\alpha_j\, f_t(G) + q_j\big) > (n-6) f_t(G) + 3s\, f_t(G).$$

This is a contradiction. Therefore,we may assume (after relabeling if necessary) that $\alpha_j\, f_t(G) + q_j > f_t(G)$ for $1 \le j \le s$. Hence by the $2t$-pebbling property, we can move at least $(k_j+1)t$ pebbles to (v_j, y) in G_j for $1 \le j \le s$. For $j > s$, we can move at least $k_j\, t$ pebbles to (v_j, y) in G_j. By the above pebbling moves, we see that at least $\sum_{j=1}^{s}(k_j+1)t + \sum_{j=s+1}^{n-3} k_j t = nt \ge f_t(F_n)$ pebbles can be moved to the copy $F_n \times \{y\}$ of F_n and so we are done (note that $(v_{n-1}, y) \in F_n \times \{y\}$). ■

Corollary 3.22. *Let F_n be a fan graph on n vertices. Then*

$$f_t(F_m \times F_n) \le f(F_m)\, f_t(F_n) \quad \textit{for all } t.$$

Corollary 3.23. *Let F_n be a fan graph on n vertices and P_m be a path on m vertices. Then*

$$f_t(F_n \times P_m) \le f(F_n)\, f_t(P_m) \quad \textit{for all } t.$$

Corollary 3.24. *Let F_n be a fan graph on n vertices and K_m be a complete graph on m vertices. Then*

$$f_t(F_n \times K_m) \le f(F_n)\, f_t(K_m) \quad \textit{for all } t.$$

Corollary 3.25. *Let F_n be a fan graph on n vertices and $K_{1,m}$ be an m-star $(m > 1)$. Then*

$$f_t(F_n \times K_{1,m}) \le f(F_n)\, f_t(K_{1,m}) \quad \textit{for all } t.$$

Corollary 3.26. *Let F_n be a fan graph on n vertices and $K_{s_1,2}$ be a complete graph with $s_1 \geq 2$. Then*

$$f_t(F_n \times K_{s_1,2}) \leq f(F_n)\, f_t(K_{s_1,2}) \quad \text{for all } t.$$

Corollary 3.27. *Let F_n be a fan graph on n vertices and $K^*_{s_1,s_2,\ldots,s_r}$ be a complete r-partite graph with s_1, s_2, ... , s_r vertices in vertex classes C_1, C_2, ... , C_r respectively. Then*

$$f_t(F_n \times K^*_{s_1,s_2,\ldots,s_r}) \leq f(F_n)\, f_t(K^*_{s_1,s_2,\ldots,s_r}) \quad \text{for all } t.$$

We now show that Conjecture 3.3 holds for the product of wheel graphs.

Theorem 3.28. *Let W_n be a wheel graph on n vertices. Then*

$$f_t(W_n \times W_m) \leq f(W_n)\, f_t(W_m) \quad \text{for all } t.$$

Proof. By Theorem 1.5 and Theorem 2.2, $f_t(W_n) = f_t(F_n)$ for all t. Since $F_n \times F_m$ is a spanning subgraph of $W_n \times W_m$, by Lemma 2.4 and Corollary 3.22, $f_t(W_n \times W_m) \leq f(W_n)\, f_t(W_m)$ for all t. ■

4 Conclusion and Future Direction

We have found the t-pebbling number of fan graphs. We have shown that fan graphs satisfy the $2t$-pebbling property. We have proved Conjecture 3.3 is true for some graphs. We have also proved that the Conjecture 3.3 is true for the product of fan graphs and for the product of wheel graphs.

Conjecture 4.1. *Conjecture 3.3 is true for a graph, which is the direct product of a tree with a tree.*

References

[1] F.R.K. Chung, *Pebbling in Hypercubes*, SIAM J. Discrete Math **2(4)** (1989), 467–472.

[2] T. A. Clarke, R. A. Hochberg, and G. H. Hurlbert, *Pebbling in diameter two graphs and products of paths*, J. Graph Th. **25** (1997), 119–128.

[3] Feng Rongquam and Ju Young Kim, *Pebbling number of some graphs*, Science in China (Series A) **43** (April 2002), no. 4, 470–478.

[4] D.S. Herscovici and A.W. Higgins, *The Pebbling Number of $C_5 \times C_5$*, Discrete Math. **187(1–3)** (1998), 123–135.

[5] G. Hurlbert, *A survey of graph pebbling*, Congressus Numerantium **139** (1999), 41–64.

[6] A. Lourdusamy, *t-pebbling the graphs of diameter two*, Acta Ciencia Indica **XXIX (M.No. 3)** (2003), 465–470.

[7] A. Lourdusamy, *t-pebbling the product of graphs*, Acta Ciencia Indica **XXXII (M.No. 1)** (2006), 171–176.

[8] A. Lourdusamy and A. Punitha, *On t-pebbling graphs*, Utilitas Mathematika, To appear.

[9] A. Lourdusamy and A. Punitha Tharani, *The t-pebbling conjecture on products of complete r-partite graphs*, Ars Combinatoria, To appear.

[10] A. Lourdusamy and S. Somasundaram, *The t-pebbling number of graphs*, South East Asian Bulletin of Mathematics **30** (2006), 907–914.

[11] L. Patcher, H.S. Snevily, and B. Voxman, *On Pebbling Graphs*, Congr. Numer. **107** (1995), 65–80.

[12] S.S. Wang, *Pebbling and Graham's Conjecture*, Discrete Math. **226(1–3)** (2001), 431–438.

[13] C. Xavier and A. Lourdusamy, *Pebbling number in graphs*, Pure and Applied Mathematika Sciences **XLIII** (March 1996), no. 1-2, 73–79.

Received: December, 2008