The Natural Partial Orders
on wpp Semigroups

Haijun Liu

Department of Mathematics
Jiangxi Normal University, Nanchang
Jiangxi 330022, P.R. China
haijunliu123@163.com

Xiaojiang Guo

Department of Mathematics
Jiangxi Normal University, Nanchang
Jiangxi 330022, P.R. China
xjguo1967@sohu.com

Shuming Qiu

Jiangxi Normal University
Nanchang, Jiangxi 330022, P.R. China

Abstract

In this paper, we investigate a kind of partial orders on wpp semigroups, which are the very generalizations of ones studied by Lawson [\textit{Proc. Edinburgh Math. Soc.}, 30(1987), 169-186]. After giving some characterizations and properties, we determine when the partial orders are compatible with respect to the multiplication.

Mathematics Subject Classification: 20M10

Keywords: wpp semigroups; Natural partial order

\footnote{This work is supported by the NSF of Jiangxi Province; the Graduate Innovation Special Foundation of Education Department of Jiangxi Province, China, no. YC08A044; and the SF of Education Department of Jiangxi Province and the SF of Jiangxi Normal University, China.}
1 Introduction

Natural partial order is of vital importance in semigroup theory. Many authors have been studying several types of natural orders. For example, Abian [1], Burgess [12], Cussman [4] and Mitsch [3]. In 1980, Nambooripad [7] investigated the natural partial orders on regular semigroups and proved that for a regular semigroup, the natural partial order \( \leq \) is compatible with respect to the multiplication if and only if it is a locally inverse semigroup (such a semigroup is a regular semigroup \( S \) in which \( eSe \) is an inverse subsemigroup for all \( e^2 = e \in S \)). Later on, Blyth and Gomes [11] considered regular semigroups in which the natural partial order is one-side compatible for the multiplication.

In 1981, Fountain [5] introduced abundant semigroups. Such semigroups are generalizations of regular semigroups. After then, by a similar method as Nambooripad, Lawson [10] defined the natural partial order on an abundant semigroup and showed that the order is compatible with the multiplication on a concordant semigroup just when the semigroup is locally type A. Furthermore, Guo and Luo [14] extended the results of Nambooripad and of Blyth and Gomes on the natural partial orders for regular semigroups to abundant semigroups, and of course those of Lawson in [10]. In this direction, Wang, Ren and Ding [2] studied the natural partial orders on \( U \)-semiabundant semigroups.

To study the larger class of semigroups than the class of abundant semigroups, Tang [13] defined the Green’s \( \ast \ast \)-relations \( L \ast \ast \) and \( R \ast \ast \). By using these Green’s \( \ast \ast \)-relations, we introduce so-called wpp semigroups, which are common generalizations of regular semigroups and abundant semigroups. It is a natural problem how to define the natural partial orders on wpp semigroups. This is the main aim of this paper.

We shall proceed as follows: after giving some known results used frequently in the sequel, we introduce a new type of partial orders on wpp semigroups in Section 3. Meanwhile, we also give some characterizations and properties of this kind of partial order; then we consider the relationship between the partial order of this paper and the one introduced by Lawson and explicitly point out the partial orders are generalizations of ones defined by Lawson [10], and they are equivalent on abundant semigroups. Finally, we determine when the partial orders are (left, right) compatible with respect to the semigroup multiplication.

2 Preliminaries

Throughout this paper, we will use notions and notations in [13]. For other terminologies not given, the reader is referred to Fountain [5] and Howie [6].

In 1951, J.A. Green defined so-called Green’s relations \( \mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D} \) and \( \mathcal{J} \), which play an important role in the study of regular semigroups. Later on,
Fountain introduced Green $*$-relations $L^*$, $R^*$, $H^*$, $D^*$ and $J^*$. These relations play an essential role in studying abundant semigroups. To generalize Green’s $*$-relations, Tang [13] introduced so-called Green $**$-relations $L^{**}$ and $R^{**}$. Following Tang [13], we define the relations on a semigroup $S$ as follows: for $a, b \in S$,

$$L^{**} = \{(a, b) \in S \times S : (\forall x, y \in S^1) axRyay \Leftrightarrow bxRby\}$$

and

$$R^{**} = \{(a, b) \in S \times S : (\forall x, y \in S^1) xaLyay \Leftrightarrow xbLyb\}.$$  

Evidently, $L^{**}$ is a right congruence and $R^{**}$ is a left congruence. It is clear that $L \subseteq L^* \subseteq L^{**}$ and $R \subseteq R^* \subseteq R^{**}$. Also, we define $H^{**} = L^{**} \cap R^{**}$ and $D^{**} = L^{**} \lor R^{**}$. These relations are usually called Green’s $**$-relations.

For convenience, we use $a^*$ to denote the typical idempotents $L^{**}$-related to $a$ while $a^\dagger$ those $R^{**}$-related to $a$. If $K^{**}$ is one of Green’s $**$-relations $L^{**}, R^{**}, H^{**}$ and $D^{**}$, we denote by $K_a^{**}$ the $K^{**}$-class of $S$ containing $a$. And, $E(S)$ denotes the set of idempotents of $S$ and $\text{Reg}(S)$ denotes the set of regular elements of $S$.

**Definition 2.1** Let $S$ be a semigroup. $S$ is called a **wpp semigroup** if

1. For all $a \in S$, $L_a^{**} \cap E(S) \neq \emptyset$ and $R_a^{**} \cap E(S) \neq \emptyset$.

2. For all $e \in L^*_a \cap E(S)$ and $f \in R^*_a \cap E(S)$, $ae = a$ and $fa = a$.

As in [16], we call a wpp semigroup an **inverse wpp semigroup** if $E(S)$ is a semilattice. It is easy to see that both inverse and adequate semigroups are inverse wpp semigroups.

**Lemma 2.2** Let $S$ be a wpp semigroup.

1. Each $L^{**}$-class and each $R^{**}$-class of $S$ contains precisely one regular $L$-class and $R$-class, respectively.

2. If $a \in S$ and $e \in E(S)$, then $ae = a \Leftrightarrow a^*e = a^*$ and $ea = a \Leftrightarrow ea^\dagger = a^\dagger$.

3. For all $a, b \in S$, we have $(ab)^*b^* = (ab)^*$ and $a^\dagger(ab) = (ab)^\dagger$. Moreover, if $S$ is also an inverse wpp semigroup, then $(ab)^\dagger\omega a^\dagger$ and $(ab)^*\omega b^*$.

**Proof.** (1) We prove only the case for $L^{**}$ because the case for $R^{**}$ may be dually proved. Let $S$ be a wpp semigroup and $e, f \in E(S)$. Now, we need only to show that if $eL^{**}f$, then $eLf$. Indeed, by the definition of wpp semigroups, $e = ef$ and $f = fe$, thereby $eLf$, as required.

(2) We only prove $ae = a \Leftrightarrow a^*e = a^*$ and the other can be dually obtained. Notice that $aa^* = a$ and so $a^*e = a^* \Rightarrow ae = a$ is obvious. So, it suffices to show the reverse implication. Now assume $ae = a$. Clearly, $aeRa$, hence $a^*eRa^*$
since $a\mathcal{L}^{**}a^*$, thereby $a^*ex = a^*$ for some $x \in S^1$, by routine computing, this shows that $exa^* \in E(S)$. Consider
\[ exa^* \cdot a^* = exa^* \text{ and } a^* \cdot exa^* = a^*, \]
we have $exa^*L^*a^*$, clearly $exa^*\mathcal{L}^{**}a^*$, therefore $exa^*e\mathcal{L}^{**}ae$ since $\mathcal{L}^{**}$ is a right congruence, now $exa^*e \mathcal{L}^{**}a$ by $ae = a$, and whence $exa^*e \mathcal{L}^{**}a^*$. On the other hand, we have $(exa^*)^2 = (exa^*)(exa^*) = ex(a^*exa^*)e = exa^*e \in E(S)$. Therefore, by (1), we can remove the double stars from $\mathcal{L}^{**}$ and obtain $exa^*e \mathcal{L} a^*$, which derives $a^* \cdot exa^* = a^*$. Also, $a^*exa^*e = a^*exa^* \cdot e = a^*e$ by $a^*exa^* = a^*$. Consequently, $a^*e = a^*$, as required.

(3) Since $abb^* = ab$, $a^1ab = ab$ and by (2), we may complete the proof of the prior half. The rest is trivial. \quad \Box

A left ideal $I$ of $S$ is called a left $**$-ideal if for all $x \in I$, $L^{**}_x \subseteq I$. Right $**$-ideal is dually defined. An ideal $I$ of $S$ is called a $*$-ideal if $I$ is not only a left $**$-ideal but also a right $**$-ideal. We shall denote by $L^{**}(a)$ [resp. $R^{**}(a)$ and $J^{**}(a)$] the smallest left [resp. right and double sides] $**$-ideal containing $a$.

**Lemma 2.3** Let $S$ be a wpp semigroup and $e \in E(S)$. Then $Se$ and $eS$ are left and right $**$-ideal, respectively. Moreover, $L^{**}(e) = Se$ and $R^{**}(e) = eS$.

**Proof.** We only prove the case for $Se$ since the other is dual. Assume $a \in Se$, then $ae = a$ and by Lemma 2.2(2), $a^*e = a^*$. If $b \in L^{**}_a$, then $b = ba^* = ba^*e$ and by Lemma 2.2(2), $b^*a^*e = b^*a^*$. But $a^*\mathcal{L}^{**}b^*$, we have $b^*a^* = b^*$ and $b^*e = b^*a^*e = b^*a^* = b^*$. Again by Lemma 2.2(2), this implies $be = b$, hence $b \in Se$, thereby $L^{**}_a \subseteq Se$ and whence $Se$ is indeed a left $**$-ideal.

For the rest part, we observe that $L^{**}(e)$ is a left ideal containing $e$, hence $Se \subseteq L^{**}(e)$. According to the above discussion, $Se$ is also a left $**$-ideal, so we have $L^{**}(e) \subseteq Se$. Consequently, $L^{**}(e) = Se$. \quad \Box

**Lemma 2.4** [8] Let $S$ be a wpp semigroup and $x, y \in S$. Then

(1) $x\mathcal{L}^{**}y$ if and only if $L^{**}(x) = L^{**}(y)$.

(2) $x\mathcal{R}^{**}y$ if and only if $R^{**}(x) = R^{**}(y)$.

Based on Lemmas 2.3 and 2.4, the following corollary is straight.

**Corollary 2.5** Let $S$ be a semigroup. Then $S$ is a wpp semigroup if and only if for any $a \in S$, $L^{**}(a) = Se$ and $R^{**}(a) = fS$ for some $e, f \in E(S)$.

Following Fountain [5], a semigroup is called abundant if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains an idempotent. In the same reference, Fountain pointed out that a semigroup $S$ is abundant if and only if for any $a \in S$, there exists $e, f \in E(S)$ such that $L^*(a) = Se$ and $R^*(a) = fS$. Based on these arguments and by Corollary 2.5, we have immediately.
Lemma 2.6 Let $S$ be a wpp semigroup. If $S$ is still abundant, then in $S$, $L^{**} = L^*$ and $R^{**} = R^*$.

A band $B$ is called right regular [resp. right normal, right quasi-normal, normal] if it satisfies the identity $xy = yx$ [resp. $xyz = yxz, xyz = xzy$, $xyzx = xzyx$] (for bands, see [9]). A semigroup $S$ is called locally $\mathcal{P}$ semigroup, if the local submonoid $eSe$ is a semigroup with the property $\mathcal{P}$, for each $e \in E(S)$.

3 The natural partial orders

In this section we shall investigate the natural partial orders on wpp semigroups.

Definition 3.1 Let $S$ be a wpp semigroup. For any $a, b \in S$, define

1. $a \leq_{w\ell} b$ if for some $e \in E(L_a^{**})$ and $f \in E(L_b^{**})$ with $e\omega f$, $a = be$, where $e\omega f$ if and only if $e = ef = fe$.
2. $a \leq_{wr} b$ if for some $e \in E(R_a^{**})$ and $f \in E(R_b^{**})$ with $e\omega f$, $a = eb$.
3. $\leq_w = \leq_{w\ell} \cap \leq_{wr}$.

Proposition 3.2 Let $S$ be a wpp semigroup. Then $\leq_{w\ell}$, $\leq_{wr}$ and $\leq_w$ are all partial orders which coincide with $\omega$ on $E(S)$, where $\omega$ is a natural partial order on $E(S)$.

Proof. We prove only the case: $\leq_{wr}$, because the other cases can be similarly proved.

The reflexivity of $\leq_{wr}$ is obvious because $S$ is a wpp semigroup.

Now suppose that $a \leq_{wr} b$ and $b \leq_{wr} a$, then there exist $e, g \in E(R_a^{**})$ and $f, h \in E(R_b^{**})$ such that $e\omega f$, $h\omega g$ and $a = eb$, $b = ha$. By Lemma 2.2 (1), $eRg$ and $eg = g$. Thus $a = eha = egha = gha = ha = b$ by $h\omega g$ giving $gh = h$. This means that $\leq_{wr}$ is anti-symmetric.

To verify that $\leq_{wr}$ is an order, it remains to show that $\leq_{wr}$ is transitive. For this, let $a \leq_{wr} b$ and $b \leq_{wr} c$, we have idempotents $e \in R_a^{**}$, $f \in R_b^{**}$ such that $e\omega f$ and $a = eb$. Also, there exist idempotents $g \in R_b^{**}, h \in R_c^{**}$ such that $g\omega h$ and $b = gc$, hence $a = egc$. By Lemma 2.2(1), $fRg$ and $ge = gfe = fe = e$, which implies $eg \in E(S)$. Notice that $ege = e$ and $eege = eg$, we have $ege \in E(S)$. On the other hand, by $g\omega h$, we have $eg \cdot h = eg$ and $h \cdot eg = hgeg = hg \cdot eg = geg = eg$ (by $ge = e$), that is, $eg\omega h$. Now, we have proved that $eg$ is an idempotent satisfying the conditions: $eg \in R_a^{**}$, $a = egc$ and $eg\omega h$ for $h \in R_c^{**}$, and whence $a \leq_{wr} c$. This shows that $\leq_{wr}$ is transitive, as required.
If $e, f \in E(S)$ and $e \leq_{wr} f$, then $g \omega h$, $e = gf$, for some $g \in R_e^{**}$ and $h \in R_f^{**}$. Thus $e = gf = gff = ef$ and $fe = fgf = hgf = gf = e$, that is, $\epsilon \omega f$, whence $\leq_{wr} |E(S) \subseteq \omega$. On the other side, if $\epsilon \omega f$, then $e = ef$, which implies $e \leq_{wr} f$, hence $\omega \subseteq \leq_{wr} |E(S)$. So $\leq_{wr}$ coincides with $\omega$ on $E(S)$. This completes the proof. 

\textbf{Theorem 3.3} Let $S$ be a wpp semigroup and $a, b \in S$. Then the following statements are equivalent:

1. $a \leq_{wr} b$.
2. For any $b^\dagger \in R_b^{**}$, there exists $a^\dagger \in R_a^{**}$ such that $a^\dagger \omega b^\dagger$ and $a = a^\dagger b$.
3. For all $b^\dagger \in R_b^{**}$, there exists $f \in E(S)$ such that $f \omega b^\dagger$ and $a = fb$.

\textbf{Proof.} 

1 $\Rightarrow$ 2 Assume $a \leq_{wr} b$, then there exist idempotents $g \in R_a^{**}$ and $e \in R_b^{**}$ such that $g \omega e$ and $a = gb$. Now let any idempotent $f \in R_b^{**}$, we have $f \mathcal{R} e$. Since $g \omega e$ and $fg = feg = eg = g$, hence $gf \in E(S)$ and $gf \omega f$. We also discover $(gf)g = g$ and $g(gf) = gf$, thus $gf \mathcal{R} g$, that is $gf \in R_a^{**}$.

Consequently, for each idempotent $f \in R_b^{**}$, there exists idempotent $gf \in R_a^{**}$ so that $gf \omega f$ and $(gf)b = g(fb) = gb = a$, which implies $gf$ is the required one.

2 $\Rightarrow$ 1 is obvious.

2 $\Rightarrow$ 3 It is trivial.

3 $\Rightarrow$ 2 We merely explain $f \in R_a^{**}$. Assume that for any $x, y \in S^1$, $xa \mathcal{L} ya$, then since $a = fb$, we have $xfb \mathcal{L} yfb$, and hence by $b \mathcal{R}^{**} b^\dagger$, we have $xfb^\dagger \mathcal{L} yfb^\dagger$. By $f \omega b^\dagger$, we know $xf \mathcal{L} yf$. Conversely, if $xf \mathcal{L} yf$, then $xfb\mathcal{L} yfb$, since $\mathcal{L}$ is right congruence. Hence $xa \mathcal{L} ya$. Consequently, $f \mathcal{R}^{**} a$ as required.

\textbf{Theorem 3.4} Let $S$ be a wpp semigroup and $x, y \in S$.

1. $x \leq_{wr} y$ if and only if $R^{**}(x) \subseteq R^{**}(y)$ and $x = fy$ for some $f \in E(R_x^{**})$.
2. $x \leq_{wr} y$ if and only if $L^{**}(x) \subseteq L^{**}(y)$ and $x = hy$ for some $h \in E(L_x^{**})$.

\textbf{Proof.} We here only prove (1). Assume $x \leq_{wr} y$. Then by Theorem 3.3, for any $y^\dagger \in R_y^{**}$, there exists $x^\dagger \in R_x^{**}$ such that $x^\dagger \omega y^\dagger$ and $x = x^\dagger y$. Hence $x = x^\dagger y = y^\dagger x^\dagger y$ and so $R^{**}(x) = R^{**}(x^\dagger) = R^{**}(y^\dagger) \subseteq R^{**}(y^\dagger) = R^{**}(y)$.

Conversely, suppose that $R^{**}(x) \subseteq R^{**}(y)$ and $x = fy$ for some idempotent $f \in R_x^{**}$. If $g \in E(R_y^{**})$, then by Lemma 2.4, $R^{**}(f) = R^{**}(x) \subseteq R^{**}(y) = R^{**}(g)$. By Lemma 2.3, $fS \subseteq gS$, so that $gf = f$, which implies $fg \mathcal{R} f$, thereby $fg \mathcal{R}^{**} f \mathcal{R}^{**} x$. Also, $x = fy = fgy$. Thus $x \leq_{wr} y$. We complete the proof.

\textbf{Corollary 3.5} Let $S$ be a wpp semigroup and $x, y \in S$. Then the following statements are equivalent:
(1) \( x \leq_w y \).
(2) There exist \( x^\dagger, x^* \) and \( y^\dagger, y^* \) such that \( x^\dagger \omega y^\dagger, x^* \omega y^* \) and \( x = x^\dagger y = yx^* \).
(3) For some \( e, f \in E(S) \), \( x = ey = yf \).

**Proof.** (1) \( \leftrightarrow \) (2) follows from Theorem 3.3 and its dual. And, according to Lemma 2.2 and Theorem 3.4, (3) \( \leftrightarrow \) (1) can be derived by using the similar description as in [10].

Now we can give some properties of these partial orders as follows.

**Proposition 3.6** Let \( S \) be a wpp semigroup.

(1) If \( x \leq_w e \) (where \( e \in E(S) \)), then \( x \in E(S) \).

(2) If \( x \leq_w y \) (where \( y \in \text{Reg}(S) \)), then \( x \in \text{Reg}(S) \).

(3) For all \( x, y \in S \), if \( x \leq_w y \) (and \( x \leq_w y \) and \( xR^{**}y \) (\( xL^{**}y \)), then \( x = y \).

**Proof.** (1) If \( x \leq_w e \), then for some idempotents \( f \in R_x^{**} \) and \( h \in R_e^{**} \) with \( f \omega h \), \( x = fe \). Note that \( ef = ehf = hfe = f \), we have \( fe \in E(S) \), that is, \( x \in E(S) \).

(2) Let \( z \) be an inverse element of \( y \), then \( yz \in E(S) \) and \( yzR_y \). Because \( x \leq_w y \), we have \( e \in E(R_y^{**}) \) and \( ewyz \) such that \( x = ey \). Hence \( xzx = (eyze)y = ey = x \), so \( x \in \text{Reg}(S) \).

(3) If \( x \leq_w y \), then \( x = ey \), for some \( e \in E(R_y^{**}) \). But \( xR^{**}y \), now \( eR^{**}y \) and \( x = ey = y \).

## 4 Compatibility with Multiplication

A wpp semigroup \( S \) is said to satisfy that \( \leq_w \) is right (left) compatible with multiplication if for all \( a, b, c \in S \), \( a \leq_w b \) implies \( ac \leq_w bc \) (\( ca \leq_w cb \)). And, \( S \) is said to satisfy that \( \leq_w \) is compatible with multiplication if \( S \) satisfies that \( \leq_w \) is right and left compatible with multiplication. The main aim of this section is to discuss the compatibility of \( \leq_w \) with multiplication for wpp semigroups.

**Definition 4.1** A wpp semigroup \( S \) is called **right ample** if for any \( x \in S \) and for any \( x^\dagger \), there exists an idempotent \( f \in \omega(x^\dagger) \) such that \( ex = xf \), for all \( e \in \omega(x^\dagger) \). **Left ample wpp semigroups** are dually defined. And, \( S \) is called **ample** if it is both left and right ample.

**Proposition 4.2** Let \( S \) be a wpp semigroup. Then \( S \) is right ample if and only if \( \leq_{wr} \leq_w \).

**Proof.** Assume that \( S \) is right ample. Clearly, \( \leq_w \subseteq \leq_{wr} \). If \( a, b \in S \) with \( a \leq_{wr} b \), then for each \( b^\dagger \) there exists \( a^\dagger \in \omega(b^\dagger) \) such that \( a = a^\dagger b \). Since \( S \) is right ample, then there exists an idempotent \( f \in \omega(b^\dagger) \) so that \( a = a^\dagger b = bf \).

Now, by Corollary 3.5, \( \leq_{wr} \subseteq \leq_w \). Thus \( \leq_{wr} = \leq_w \).
Conversely, suppose that $\leq_{wr}=\leq_w$. For each $a \in S$ and for all $a^\dagger$, if $e \in \omega(a^\dagger)$, then $a R^{**} a^\dagger$ implies $eaR^{**} a^\dagger = e$. Thus by Definition 2.1, $ea \leq_{wr} a$ and $ea \leq_w a$, thereby by Corollary 3.5, there exists an idempotent $f \in \omega(a^*)$ such that $ea = af$. Consequently, $S$ is right ample.

By Proposition 4.2 and its dual, we immediately have

**Corollary 4.3** Let $S$ be a wpp semigroup. Then $S$ is ample if and only if $\leq_{wr}=\leq_w$.

**Lemma 4.4** Let $S$ be a wpp semigroup. If $\leq_w$ is right compatible with multiplication of $S$, then $S$ is right ample.

**Proof.** Let $a \in S$. For all $x \in \omega(a^\dagger)$, then $x \leq_w a^\dagger$ so that $xa \leq_w a^\dagger a = a$, thereby by Corollary 3.5, there exists $f \in \omega(a^*)$ such that $xa = af$. Therefore, $S$ is right ample. $\Box$

The following corollary follows from Lemma 4.4 and its dual.

**Corollary 4.5** Let $S$ be a wpp semigroup. If $\leq_w$ is compatible with the multiplication of $S$, then $S$ is ample.

**Lemma 4.6** Let $S$ be a wpp semigroup and $e \in E(S)$. Then $eSe$ is a wpp submonoid of $S$. Moreover, $L^{**}eSe = L^{**}|eSe$ and $R^{**}eSe = R^{**}|eSe$.

**Proof.** If $a \in eSe$, then $ae = a$ and by Lemma 2.2, $fe = f$ for any $f \in E(L_a^*)$, thereby $ef \in E(S)$, $ef Ae$ and $ef Lf$. But $L \subseteq L^{**}$, now $ef L^{**}f$, and by $f L^{**}a$, we have $ef L^{**}a$. This can show that $ae = f = a$, and also $ef = efe \in E(eSe)$. Now, we have $ef L^{**}(eSe)a$, by noting that $L^{**}eSe \subseteq L^{**}eSe$. Consequently, again together with the dual, $eSe$ is wpp while $L^{**}eSe = L^{**}|eSe$ and dually $R^{**}eSe = R^{**}|eSe$. We complete the proof. $\Box$

A semigroup $S$ is called $L^{**}[R^{**}]-unipotent$ if its idempotents form a right [left] regular band. We claim that each $L^{**}[R^{**}]-class$ of an $L^{**}[R^{**}]-unipotent$ wpp semigroup contains exactly one idempotent. Indeed, if $S$ is an $L^{**}-unipotent$ wpp semigroup, and $e,f \in E(S)$ with $eL^{**}f$, then by Lemma 2.2(1), $eL^{**}f$ and $e = ef = fef = ff = f$ since $E(S)$ is a right regular band.

Now we arrive at the main results of this section.

**Theorem 4.7** Let $S$ be a right ample wpp semigroup. Then the following are equivalent:

1. $S$ is a locally $L^{**}-unipotent$ semigroup.
2. $\leq_w$ is right compatible with the multiplication of $S$.
3. For all $e \in E(S)$, $E(eS)$ forms a right regular band under semigroup multiplication.
4. For all $e \in E(S)$, $E(Se)$ forms a right quasi-normal band under semigroup multiplication.
5. For all $e,f \in E(S)$, $E(eSf)$ forms a right regular band under semigroup multiplication.
Proof. We shall prove this theorem by proving (1) ⇒ (3) ⇒ (2) ⇒ (1), (1) ⇒ (4) ⇒ (1) and (3) ⇒ (5) ⇒ (1). It suffices to verify (3) ⇒ (2) because others can be obtained by using similar descriptions as in [14, Theorem 5.5].

Now, assume (3) holds. If \(a, b \in S\) with \(a \leq_{w} b\), then for each \(b^{\dagger}\), there exists \(a^{\dagger} \in \omega(b^{\dagger})\) such that \(a = a^{\dagger}b\), and clearly \(ac = a^{\dagger}bc\) for any \(c \in S\). On the other hand, by Lemma 2.2(3), \(b^{\dagger}(bc)^{\dagger} = (bc)^{\dagger}\), thereby \((bc)^{\dagger}b^{\dagger} \in E(S)\) and \((bc)^{\dagger}R(bc)^{\dagger}b^{\dagger}\), obviously \(bcR^{\ast\ast}(bc)^{\dagger}b^{\dagger}\). Note that \(a^{\dagger}, (bc)^{\dagger}b^{\dagger} \in E(b^{\dagger}Sb^{\dagger})\) and since \(E(eS)\) is a right regular band, \(a^{\dagger}(bc)^{\dagger}b^{\dagger} \in E(b^{\dagger}Sb^{\dagger})\) and

\[(bc)^{\dagger}b^{\dagger} \cdot a^{\dagger}(bc)^{\dagger}b^{\dagger} = (bc)^{\dagger}b^{\dagger} \cdot a^{\dagger} \cdot (bc)^{\dagger}b^{\dagger} = a^{\dagger}(bc)^{\dagger}b^{\dagger},\]

thus \(a^{\dagger}(bc)^{\dagger}b^{\dagger} \in \omega((bc)^{\dagger}b^{\dagger})\). We have now proved that \(ac = a^{\dagger}bcR^{\ast\ast}a^{\dagger}(bc)^{\dagger}b^{\dagger}\) and \(ac = (a^{\dagger}(bc)^{\dagger}b^{\dagger})(bc)\), that is, \(ac \leq_{wr} bc\), and whence \(ac \leq_{w} bc\) by Proposition 4.2. We complete the proof. \(\square\)

Notice that in the variety of bands, the intersection of a right regular band and a left quasi-normal band is a right normal band, the intersection of a left regular and a right quasi-normal band is a left normal band, and the intersection of a right normal (regular) band and a left normal (regular) band is a semilattice (for more details, see [9, Chapter II]). If we combine Theorem 4.7 with its dual, we can immediately obtain the corollary below.

**Corollary 4.8** If \(S\) is an ample wpp semigroup. Then the following statements are equivalent:

1. \(S\) is a locally inverse wpp semigroup.
2. \(\leq_{w}\) is compatible with the multiplication of \(S\).
3. For all \(e \in E(S), E(eS)\) is a right normal band under semigroup multiplication.
4. For all \(e \in E(S), E(Se)\) is a left normal band under semigroup multiplication.
5. For all \(e, f \in E(S), E(eSf)\) is a semilattice under semigroup multiplication.

**Acknowledgment:** The authors would like to thank Dr. Emil Minchev and the referees for the publication of this paper.

**References**


Received: November, 2008