Introducing an Iterative Method for
Solving a Special FDE

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Abstract
In this paper, we introduce an iterative method for solving fractional differential equation

$$D^\alpha[p(x)y'(x)] + \lambda q(x)y(x) = g(x), \quad 0 < \alpha \leq 1, \quad x \in (0,a)$$

with boundary conditions $y'(0) = 0$, $y(a) = 0$, further we prove uniqueness of solution. Then with this method, we shall solve fractional Diffusion-Wave equation.

Keywords: Fractional differential equation (FDE); Iterative method; Diffusion-Wave equation

1 Introduction
The method of solution to solve Eq(1) based on an efficient technique named iterative approximation method. Similar to this method, Adomian decomposition method, has been used by authors in order to solve several types of differential equations. See for example Adomian [1], Al-Mdallal [2], Wazwaz[7] and El-Wakil and Abdou[5]. In this paper, we investigate the following boundary value problem

$$D^\alpha[p(x)y'(x)] + \lambda q(x)y(x) = g(x), \quad 0 < \alpha \leq 1, \quad x \in (0,a)$$

subject to

$$y'(0) = 0, \quad y(a) = 0$$
where \( p(x), q(x) \) are positive and smooth functions. In Eq(1), if \( a = 1 \) and \( g(x) = 0 \), then Eq(1) is named fractional Sturm-Liouville problem [2]. This paper is divided in three sections as follows: Analysis of the iterative method, proving uniqueness of solution and in the last section examples and results will be given.

2 Analysis of the iterative method

**Definition:** The Caputo’s derivative of order \( \alpha > 0 \) with \( m - 1 < \alpha \leq m \) and \( m \in \mathbb{N} \) is defined as:

\[
D^\alpha y(x) = \frac{1}{\Gamma(m - \alpha)} \int_0^x (x - t)^{m-\alpha-1} D^m y(t)dt
\]

**Lemma:** For \( k \in \mathbb{N}, \alpha \in \mathbb{R}^+ \), if \( m - 1 < \alpha \leq m \) and \( y \in L^1[a, b] \) then

\[
D^\alpha I^\alpha y(x) = y(x), \quad I^\alpha D^\alpha y(x) = y(x) - \sum_{k=0}^{m-1} y^{(k)}(0) \frac{x^k}{k!}.
\]

**proof:** [6].

We consider equation (1), by using the previous Lemma, we have:

\[
I^\alpha(D^\alpha[p(x)y'(x)]) + I^\alpha(\lambda q(x)y(x)) = I^\alpha[g(x)]
\]

\[
p(u)y'(u) - p(0)y'(0) + \frac{\lambda}{\Gamma(\alpha)} \int_0^u (u - t)^{\alpha-1} q(t)y(t)dt = \frac{1}{\Gamma(\alpha)} \int_0^u (u - t)^{\alpha-1} g(t)dt
\]

So, from boundary condition \( y'(0) = 0 \), we have:

\[
y'(u) + \frac{\lambda}{\Gamma(\alpha)p(u)} \int_0^u (u - t)^{\alpha-1} q(t)y(t)dt = \frac{1}{\Gamma(\alpha)} \int_0^u (u - t)^{\alpha-1} g(t)dt
\]

Now, by integration in the interval \((0, x)\) yields:

\[
y(x) = y(0) + \frac{1}{\Gamma(\alpha)} \int_0^x \int_0^u (u - t)^{\alpha-1} g(t)dtdu
\]

\[
-\frac{\lambda}{\Gamma(\alpha)} \int_0^x \frac{1}{p(u)} \int_0^u (u - t)^{\alpha-1} q(t)y(t)dtdu
\]

(2)
We consider a recursive sequence \( y_n \) of functions on \([0, a]\) and correspondingly an infinite series \( \sum u_n \) where \( u_n = y_n - y_{n-1} \). By using M-test Weierstrass, we conclude that this series is uniformly convergent to a function \( u \). Since,

\[
\sum_{n=1}^{N} u_n = \sum_{n=1}^{N} (y_n - y_{n-1}) = y_N - y_0
\]

therefore \( y_n \) tend to \( u + y_0 \) on \([0, a]\), and from uniform convergence, it follows that \( u + y_0 \) is a solution of (2), and hence a solution of (1). Now, at first we define \( y_0 \) and then \( y_n \) on \([0, a]\) by iteration. Let

\[
y_0(x) = y(0) + \frac{1}{\Gamma(\alpha)} \int_0^x \frac{1}{p(u)} \int_0^u (u - t)^{\alpha-1} g(t) dt du
\]

\[
y_1(x) = y_0(x) - \frac{\lambda}{\Gamma(\alpha)} \int_0^x \frac{1}{p(u)} \int_0^u (u - t)^{\alpha-1} q(t) y_0(t) dt du
\]

So, by induction we obtain :

\[
y_n(x) = y_0(x) - \frac{\lambda}{\Gamma(\alpha)} \int_0^x \frac{1}{p(u)} \int_0^u (u - t)^{\alpha-1} q(t) y_{n-1}(t) dt du.
\]

Now, we notice that

\[
|y_0(x)| \leq |y(0)| + \frac{1}{\Gamma(\alpha)} \int_0^x \frac{1}{p(u)} \int_0^u (u - t)^{\alpha-1} |g(t)| dt du \leq A
\]

and \( \exists M, N > 0 \),

\[
\left| \frac{\lambda}{\Gamma(\alpha)p(x)} \right| \leq N, \quad \left| \int_0^u (u - t)^{\alpha-1} g(t) dt \right| \leq M,
\]

because of \( p(x) \), \( q(x) \) and \( g(x) \) are continuous and \( p(x) \) is nonzero on \([0, a]\). Now, we define \( u_n = y_n - y_{n-1} \). consequently

\[
|u_1(x)| = |y_1(x) - y_0(x)| \leq M N A x
\]

and, by induction, if

\[
|u_n(x)| = |y_n(x) - y_{n-1}(x)| \leq \frac{M^n N^n A x^n}{n!}
\]

then

\[
|u_{n+1}(x)| = |y_{n+1}(x) - y_n(x)| \leq \frac{M^{n+1} N^{n+1} A x^{n+1}}{(n + 1)!}
\]
Having inductively found bounds for all the $|y_n(x) - y_{n-1}(x)|$, over $[0, a]$ we can now define the nonnegative constants $E_n$ as follows:

$$|u_n(x)| = |y_n(x) - y_{n-1}(x)| \leq \frac{M^n N^n A x^n}{n!} \leq \frac{M^n N^n A a^n}{n!} = E_n$$

for $n \geq 1$. Now, and consequently

$$\sum_{n=1}^{\infty} E_n = A \sum_{n=1}^{\infty} \frac{(MNa)^n}{n!} = A(e^{MNa}) - 1,$$

the exponential series for $e^x$ being convergent for all values of its argument $x$. So, all the hypotheses for the application of the Weierstrass M-test [4] are satisfied and we can deduce that Since,

$$\sum_{n=1}^{\infty} (y_n - y_{n-1})$$

is uniformly convergent on $[0, a]$, to a function $u$. Then, as we showed above in our general discussion, the sequence $\{y_n\}$ converges uniformly to $y \equiv u + y_0$ on $[0, a]$. Since every $y_n$ is continuous on $[0, a]$, then $y$ is continuous also. So,

$$\int_0^u (u - t)^{\alpha-1} q(t)y_n(t)dt \to \int_0^u (u - t)^{\alpha-1} q(t)y(t)dt$$

and from the Lebesgue Dominated Convergence Theorem, we arrive

$$\int_0^x \frac{1}{p(u)} \int_0^u (u - t)^{\alpha-1} q(t)y_n(t)dtdu \to \int_0^x \frac{1}{p(u)} \int_0^u (u - t)^{\alpha-1} q(t)y(t)dtdu.$$ 

Therefore, we conclude that $y$ is the solution of the integral equation (2).

## 3 Uniqueness Of Solution

We suppose $Y = Y(x)$ is of another solution integral equation (1). The continuous function $y - Y$ is bounded on $[0, a]$. Suppose that

$$|y(x) - Y(x)| \leq p$$

for all $x \in [0, a]$. Inductively, we can show that:

$$|y(x) - Y(x)| \leq \frac{(NM a)^n p}{n!}$$

since the right hand side of the inequality tends to zero as $n \to \infty$ then

$$y(x) = Y(x)$$

for all $x \in [0, a]$. 
4 Examples

Example 1: We consider the fractional differential equation

\[ D^{\frac{1}{2}} y'(x) + \lambda y(x) = x, \quad x \in (0, a) \tag{3} \]

with boundary conditions, \( y'(0) = 0 \), \( y(a) = 0 \). So, from equation (2) we have

\[ y(x) = y(0) + \frac{1}{\Gamma(\frac{1}{2})} \int_0^x \int_0^u (u-t)^{-\frac{1}{2}} t dt du - \frac{\lambda}{\Gamma(\frac{1}{2})} \int_0^x \int_0^u (u-t)^{-\frac{1}{2}} y(t) dt du \]

Applying, iterative method we obtain

\[ y_0(x) = A + \frac{1}{\Gamma(\frac{1}{2})} x^{\frac{5}{2}}, \]

\[ y_1(x) = A + \frac{1}{\Gamma(\frac{1}{2})} x^{\frac{5}{2}} - \frac{\lambda A}{\Gamma(\frac{5}{2})} x^{\frac{3}{2}} - \frac{\lambda}{\Gamma(5)} x^4 = y_0(x) - \frac{\lambda A}{\Gamma(\frac{5}{2})} x^{\frac{3}{2}} - \frac{\lambda}{\Gamma(5)} x^4, \]

\[ y_2(x) = A + \frac{1}{\Gamma(\frac{1}{2})} x^{\frac{5}{2}} - \frac{\lambda A}{\Gamma(\frac{5}{2})} x^{\frac{3}{2}} - \frac{\lambda}{\Gamma(5)} x^4 + \frac{\lambda^2 A}{\Gamma(4)} x^3 + \frac{\lambda}{\Gamma(\frac{13}{2})} x^{\frac{11}{2}} \]

\[ = y_1(x) + \frac{\lambda^2 A}{\Gamma(4)} x^3 + \frac{\lambda}{\Gamma(\frac{13}{2})} x^{\frac{11}{2}}, \]

and, by induction

\[ y_n(x) = y_{n-1}(x) + \frac{(-\lambda)^n A}{\Gamma(\frac{3n}{2} + 1)} x^{\frac{3n}{2}} + \frac{(-\lambda)^n}{\Gamma(\frac{5}{2}(n + 1) + 2)} x^{\frac{3}{2}(n+1)+1}, \]

So,

\[ y(x) = \sum_{i=0}^{\infty} \frac{(-\lambda)^i A}{\Gamma(\frac{3i}{2} + 1)} x^{\frac{3i}{2}} + \sum_{i=0}^{\infty} \frac{(-\lambda)^i}{\Gamma(\frac{5}{2}(i + 1) + 2)} x^{\frac{3}{2}(i+1)+1} \]

is a solution of Eq(3).

Example 2: We consider the following nonhomogeneous fractional Diffusion-Wave equation

\[ D_t^\alpha \left( \frac{\partial u}{\partial t}(x, t) \right) = \frac{\partial^2 u(x, t)}{\partial x^2} + t, \quad 0 \leq \alpha \leq 1 \tag{4} \]
with the boundary conditions:

\[
\begin{align*}
  u(0, t) &= u(\pi, t) = 0, \quad t \geq 0, \\
  u(x, 0) &= f(x), \quad 0 < x < \pi, \\
  u_t(x, 0) &= 0, \quad 0 < x < \pi
\end{align*}
\]

By separation of variables method, we suppose \( u(x, t) = X(x)T(t) \) is a solution of homogeneous part (4), together with the boundary conditions. So, we have

\[
X''(x) + \lambda X(x) = 0, \quad X(0) = X(\pi) = 0
\]

(5)\]

\[
D^\alpha T'(t) + k\lambda T(t) = 0, \quad t \geq 0.
\]

(6)

The Sturm-Liouville problem (5) has eigenvalues \( \lambda_n = n^2 \) and the corresponding eigenfunctions \( X_n(x) = \sin nx, (n = 1, 2, \ldots) \). For the solution of (6), we use of iterative method with \( g(t) = 0 \). So,

\[
T_0(t) = T(0) = B, \\
T_1(t) = B - \frac{Bkn^2}{\Gamma(\alpha+2)} t^{\alpha+1}, \\
T_2(t) = B - \frac{Bkn^2}{\Gamma(\alpha+2)} t^{\alpha+1} + \frac{B(kn^2)^2}{\Gamma(2\alpha+3)} t^{2\alpha+2},
\]

\[
\vdots
\]

\[
T_m(t) = B - \frac{Bkn^2}{\Gamma(\alpha+2)} t^{\alpha+1} + \frac{B(kn^2)^2}{\Gamma(2\alpha+3)} t^{2\alpha+2} + \cdots + (-1)^n \frac{B(kn^2)^m}{\Gamma(m\alpha+m+1)} t^{m(\alpha+1)}.
\]

Consequently,

\[
T(t) = \sum_{i=0}^{\infty} \frac{B(-kn^2)^i}{\Gamma(i\alpha+i+1)} t^{i(\alpha+1)}
\]

is the solution of (6). Now we look for the solution of the nonhomogeneous problem (4) which is of the form

\[
u(x, t) = \sum_{n=1}^{\infty} H_n(t) \sin nx.
\]

From [3] we have

\[
1 = \sum_{n=1}^{\infty} \frac{2[1 - (-1)^n]}{n\pi} \sin nx, \quad 0 < x < \pi.
\]

Therefore, from (4) we conclude

\[
\sum_{n=1}^{\infty} [D^\alpha H'_n(t) + kn^2 H_n(t)] \sin nx = \sum_{n=1}^{\infty} \frac{2[1 - (-1)^n]}{n\pi} t \sin nx.
\]

(7)
Hence, we get by identifying the coefficients two side of the sin series (7)

\[ D^\alpha H_n'(t) + kn^2 H_n(t) = \frac{2[1 - (-1)^n]}{n\pi} t, \quad n = 1, 2, \ldots. \]

Now, by using iterative method and letting \( H_n(0) = B, M = \frac{2[1 - (-1)^n]}{n\pi} \), we have

\[
H_{n_0}(t) = B + \frac{M}{\Gamma(\alpha+3)} t^{\alpha+2}, \\
H_{n_1}(t) = B + \frac{M}{\Gamma(\alpha+3)} t^{\alpha+2} - \frac{B kn^2}{\Gamma(\alpha+2)} t^{\alpha+1} - \frac{M kn^2}{\Gamma(2\alpha+4)} t^{2\alpha+3}, \\
H_{n_2}(t) = B + \frac{M}{\Gamma(\alpha+3)} t^{\alpha+2} - \frac{B kn^2}{\Gamma(\alpha+2)} t^{\alpha+1} - \frac{M kn^2}{\Gamma(2\alpha+4)} t^{2\alpha+3} + \frac{B kn^2}{\Gamma(\alpha+2)} t^{2(\alpha+1)} + \frac{M (kn^2)^2}{\Gamma(3\alpha+5)} t^{3\alpha+4},
\]

\[
\vdots
\]

\[
H_{n_m}(t) = B \sum_{i=1}^{m} \frac{(-kn^2)^i}{\Gamma(i\alpha+i+1)} t^{(i+1)\alpha+i+2} + M \sum_{i=1}^{m} \frac{(-kn^2)^i}{\Gamma((i+1)\alpha+i+3)} t^{(i+1)\alpha+i+2}.
\]

Hence,

\[
H_n(t) = b_n \sum_{i=1}^{\infty} \frac{B(-kn^2)^i}{\Gamma(i\alpha+i+1)} t^{(i+1)\alpha+i+2} + M \sum_{i=1}^{\infty} \frac{(-kn^2)^i}{\Gamma((i+1)\alpha+i+3)} t^{(i+1)\alpha+i+2},
\]

where,

\[
b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nxdx
\]

(noting to condition \( u(x, 0) = f(x), \ 0 < x < \pi \)).

References


Received: October, 2008